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ON LARGE INDIVIDUALIZED AND DISTRIBUTIONALIZED EXCHANGE ECONOMIES WITH INFINITELY MANY COMMODITIES

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Abstract. We establish the existence and core equivalence theorems for exchange economies with the commodity spaces ℓ^∞ and $ca(K)$, respectively where K is a compact metric space. The keys for these results are the saturated measure space of consumers and the exact Fatou's lemma on it. We also discuss the relationship between the individualized and distributionalized formulations of a market.

1. Introduction

Aumann [2, 3] introduced a continuum of the consumers space, i.e., the unit interval $[0, 1]$, with the Lebesgue measure into general equilibrium models to make the concept of the perfect competition to be mathematically precise and rigorous. As is well known, Aumann's economic motivation was to formulate and prove the core equivalence of the competitive equilibrium that was an exact version of the limit theorem of the core, the asymptotic counterpart that was previously proven by Debreu and Scarf [9]. The concept of allocations was generalized from finite dimensional vectors to integrable maps, i.e., elements of an infinite dimensional vector space $L^1[0, 1]$. Subsequently, Hildenbrand [18] formulated the (continuum) economy as a measurable map $\mathcal{E}$ from an atomless measure space of consumers $(A, \mathcal{A}, \lambda)$ to the space of agents' characteristics $\mathcal{P} \times \Omega$, where $\mathcal{P}$ is the set of consumers' preferences and Ω is that of initial endowments. The concept of *large individualized economy* (LIE) was established and the competitive equilibrium of the LIE (*allocative equilibrium*) was also presented.

Several mathematical economists have tried to extend Aumann's existence theorem to models with infinite dimensional commodity spaces. Among those, Bewley [6], Noguchi [47], and Suzuki [61] proved the equilibrium existence theorems for the economies with the measure space of agents on the commodity space ℓ^∞. Khan–Yannelis [35] and Noguchi [46] proved the same theorem for the commodity space continuum of consumer models that is a separable Banach space with an interior point of the positive orthant. Ostroy and Zame [48] proved an equilibrium existence

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theorem on the commodity space $ca(K)$. The first main topic of this paper is to explore this issue. In particular, we prove the existence of competitive equilibria for LIEs with the commodity spaces ℓ^∞ and $ca(K)$.

We believe that our choice of the commodity spaces, ℓ^∞ and $ca(K)$, is justified in that these spaces have clear and definitive interpretations. Moreover, the models of these spaces have common mathematical characters. In particular, those authors who studied the ℓ^∞ or $ca(K)$ spaces all worked with the weak* topology on the space; hence, the resource condition in terms of the Gelfand integral. In addition, they set the price space as the predual of the commodity space. This is in contrast with [35, 46] in which the price space is simply taken as the norm dual of the commodity space.

Aumann's fundamental observation is that the convexity of the preferences is unnecessary for finite dimensional models. The first problem to generalize his result to infinite dimensional settings is that we need the convexity in the infinite dimensional settings. Indeed, all of the aforementioned authors assumed the convexity of preferences or similar assumptions[1]. The convexity assumptions obviously weaken the impact of Aumann's classical result and reveal the "convexifying effect" of large numbers of the economic agents that is a result of Fatou's lemma or Liapunov's convexity theorem, which are known to fail on the infinite dimensional spaces. Therefore, we must assume the convexity directly or determine some conditions that recover the missing convexity.

Rustichini and Yannelis [55] was probably the first paper to address the equilibrium existence problem for a model on an infinite dimensional commodity space without the convexity of preferences. They concluded that to obtain any Fatou-type theorem (lemma), one must have "many more agents than commodities." According to Mertens [45, p.189], this "many more agents than commodities" thesis is seemingly at first addressed by Aumann himself in the context of the core equivalence theorem[2]; hence we call it *Aumann's thesis.* Aumann's thesis in the sense of Rustichini–Yannelis is stated as follows. Let $\mathcal{A}_E = \{A \cap E |\ A \in \mathcal{A}\}$ be the sub-σ algebra of $\mathcal{A}$ restricted to $E \in \mathcal{A}$. Recall that for any (real) vector space, an algebraic Hamel basis exists. The cardinality of any Hamel basis of a vector space L is the same, and we denote it as $dim(L)$. Rustichini and Yannelis proposed the next condition:

(RY): For any $E \in \mathcal{A}$ with $\lambda(E) > 0$, $dim(\mathscr{L}^\infty_E(\lambda)) > dim(L)$,

where $\mathscr{L}^\infty_E(\lambda)$ is the space of essentially bounded functions on E and L is the commodity space. Note that this condition involves both consumers and commodities spaces. Podczeck criticized this condition that "one may wish to interpret an atomless measure spaces as an idealization of a large but finite number of them. From this point of view, it is preferable to keep a measure space of agents 'small' ([50], p.386)."

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[1]Noguchi assumed that a commodity vector does not belong to the convex hull of its preferred set.

[2]Aumann was indeed correct, see [51, 65].

In the present paper, we propose that Aumann's thesis is manifestly represented when we set a *saturated* (or *super-atomless*[3]) measure space of consumers. A measure space $(A, \mathcal{A}, \lambda)$ is saturated if its subalgebra $\mathcal{A}_E$ are uncountable generated for any $E \in \mathcal{A}$ with $\lambda(E) > 0$ modulo null sets. Aumann's thesis is now embodied intrinsically within the measure space of consumers rather than a condition imposed on it from outside. It is realized naturally in our models.

As Podczeck [52] indicated, the saturated measure space itself does not necessarily have an extraordinarily large cardinality. This can be explained by the nontrivial Loeb measure spaces (Keisler and Sun [21]) that are important examples of the saturated measure spaces. It is possible for some nontrivial Loeb measure spaces to have the cardinality of the continuum; hence, they can be identified with the unit interval on the real line.

On the saturated measure space, we can prove Fatou's lemma for the Gelfand integrable maps [14, 30]. This is the key result for proving the existence of equilibria in the spaces ℓ^∞ and $ca(K)$ without the convexity of preferences. Moreover, the saturation is known to be equivalent with the convexity of the Gelfand integral of correspondences [52, 57]. We use it in our proofs of core equivalence theorems for those spaces.

There is another formalism of large economies that we call the *large distributionalized economy* (LDE). The LDE was invented by Hart, Hildenbrand, and Kohlberg [16] in which the economy was defined as a probability measure μ on the space of agents' characteristics $\mathcal{P} \times \Omega$. The allocation of this economy is also defined as a probability measure ν on $X \times \mathcal{P} \times \Omega$, where X is a consumption set that is assumed to be identical among all consumers. The marginal distribution of ν on $\mathcal{P} \times \Omega$ is required to coincide with μ for consistency. We call the competitive equilibrium of the LDE the *distributive equilibrium*. Their purpose of these notions of economy and equilibria is to introduce collective or macroeconomic perspectives into the general equilibrium theory:

> If the set A of agents is small (e.g., $\#A = 3$), then the description of an economy as an assignment $a \mapsto (\succsim_a, e_a)$ of agents to characteristics seems to be the appropriate one. However, when the set A is large (e.g., $\#A = 1000$), one is tempted to give up this individualistic description and to replace it by a more collective view, namely to consider the *distribution* of the mapping $\mathcal{E}$ [16, pp. 159–160, italics by Hart–Hildebrand–Kohlberg].

They also recognized that this equilibrium concept is not suitable for core equilibrium:

> ... [I]f an economy is described by a distribution of agents' characteristics only, then it is not clear how the core of that economy should be defined. The concepts of 'coalition' and 'to improve upon' require the individualistic description of an economy as a mapping which assigns to every individual agent his characteristics [*ibid.*].

[3]Super-atomless was coined by Podczeck [52].

The LDE formalism was successfully applied to the model of the space $ca(K)$ by Mas-Colell [41] and Jones [19]. They could prove the existence of distributive equilibria without assuming the convexity of preferences because neither Fatou's lemma nor Liapunov's theorem is used in the proof for the existence of these equilibria. Mas-Colell also proved the core equivalence by invoking the (individualistic or mapping) *representation* for distributive equilibrium, where a pair of measurable mappings $\xi : A \to X$ and $\mathcal{E} : A \to \mathcal{P} \times \Omega$ is called a representation[4] of ν if $\nu = (\xi, \mathcal{E})_* \lambda \equiv \lambda \circ (\xi, \mathcal{E})^{-1}$. Their observations were further confirmed for the model of ℓ^∞ by [59].

The saturated measure space also casts light on the relationship between the allocative and distributive equilibria. When $X \times \mathcal{P} \times \Omega$ is a complete and separable metric space, the LDE μ and its distributive equilibrium ν have a representation of $\mathcal{E}$ and $(\xi, \mathcal{E}')$, respectively. We must notice that $\mathcal{E}$ and $\mathcal{E}'$ are not necessarily equal; hence, the important question is the following. Let an economy μ and its equilibrium ν be given. For each representation $\mathcal{E}$ of μ, do we have an equilibrium allocation map ξ such that $(\xi, \mathcal{E})$ represents ν? Generally, the answer is no, as suggested by Mas-Colell;

> While given an economy μ, we can prove the existence of an equilibrium distribution ν (and therefore of an equilibrium allocation $\xi : A \to X$ for *some* representation of μ, $\mathcal{E} : A \to \mathcal{P} \times \Omega$), it is unlikely that given a representation $\mathcal{E} : A \to \mathcal{P} \times \Omega$, there is an allocation $\xi : A \to X$ that is an equilibrium with respect to $\mathcal{E}$ [41, p. 273 with notations changed].

However, we obtain an affirmative answer when the measure space is saturated (Theorem 4.4). Hence, the distributive equilibrium ν does not lose any individual information when one works with the saturated economies. In addition, the allocative and the distributive equilibria are equivalent in a strong sense for economies with the saturated measure space of consumers. We discuss more the realization of the distributive equilibria in Section 4.

The paper is organized as follows. Section 2 is devoted to mathematical preliminaries. Its aim is twofold. The first is to fix our mathematical symbols and notations. The second is to collect *all* mathematical results used in the text for our readers' convenience and enable the paper to be self-contained. In particular, the definition of saturated measure spaces and a Fatou's lemma for the Gelfand integral on them that plays a key role in the proof of the existence theorems are presented. However, many of the results are standard. Hence, experts can skip this section.

In Section 3, we present our basic model that includes exchange economies with the commodity spaces ℓ^∞ and $ca(K)$. The consumption set of the basic model is assumed to contain a finite dimensional and unbounded part for relaxing the interiority assumptions on the initial endowments that are commonly assumed in the literature. Indeed, [6, 35, 46, 47, 55, 60, 61] all assumed that individual initial endowments belong to the (norm) interior of the consumption set. Obviously, such

[4]Similarly, a mapping $\mathcal{E} : A \to \mathcal{P} \times \Omega$ is a representation of μ, or $\mathcal{E}$ represents μ if $\mu = \mathcal{E}_* \lambda \equiv \lambda \circ \mathcal{E}^{-1}$. Notice that representations are not generally unique.

an assumption is very strong in the economies with a continuum of traders[5]. We dispense with the interiority assumption by making use of the monotonicity of preferences. As is well known, these remarks do not apply for models on $ca(K)$ in which all consumers are assumed to have the nonnegative orthant of $ca(K)$ as their common consumption set. See also Section 5.

In Section 4, we discuss the relationships between large individualized and distribuitionalized economies. The first issue is the realization of distributive equilibria explained previously. A distributive equilibrium of a distributionalized economy μ is said to be *realizable* if for every individualized economy $\mathcal{E}$ that represents μ, an equilibrium allocation ξ of $\mathcal{E}$ represents ν or $\nu = (\xi, \mathcal{E})_*\lambda$. As stated, Theorem 4.4 shows that distributive equilibria are realizable if the underlying measure space is saturated. This result has been already obtained and applied in specific models [31, 62]. We present and discuss it in a general framework. The second issue to be explored is the symmetricity of equilibria that was pioneered in the noncooperative game theory [32, 42]. Intuitively, a distributive equilibrium is called *symmetric* if the consumers with the same characteristics consume the same commodity bundle. We discuss the symmetric equilibria and related results.

In Section 5, we study the existence and core equivalence of allocative equilibria for economies with commodity spaces of ℓ^∞ and $ca(K)$. First, the existence and the core equivalence theorems [59] for LDEs are reformulated and translated into LIEs. Next, we show similar results for LIE with the commodity space $ca(K)$. Our fundamental observation is the following. As is well known, the basic problem concerning space $ca(K)$ is that the positive orthant has an empty norm interior. Hence, one needs some conditions to remedy this mathematical difficulty. Jones [19, 20] introduced the condition of a bounded marginal rate of substitution ((US), Assumption 5.5). This is a local condition in the sense that it restricts the substitutability between two commodities locally or commodities being sufficiently similar, and he proved the existence of equilibria (hence core) under the assumption (US). Unfortunately the local substitutability is not sufficient for the core equivalence. The references [15, 48] for instance worked with the bounded rate of substitution called (BRS) in the global sense that generates an open cone of the commodity space. This condition was originally invented by Mas-Colell [43] as the *properness*. Ostroy and Zame [48] assumed the local[6] and the global substitutability at once for both of the existence and core equivalence. Greinecker and Podczeck [15] discarded the metrizability of K and weakened conditions on the bounds for local and global substitutablilities. But they kept the basic stance of [48] and did not ask the question of existence. We clarified the roles of local and global substitutability for existence and core equivalence, respectively, namely that the local substitutability is required for existence but not for core equivalence and core equivalence can be proved without local substitutability. Specifically speaking, we will prove in Theorem 5.7 existence of an equilibrium assuming (US) without any global substitutability conditions and

[5]Precisely, Khan–Yannelis, Noguchi, and Rustichini–Yannelis assumed that there exists $z \in X$ such that $\omega - z$ belongs to the (norm) interior of X.

[6]They proposed an another version of the local substitutability called (PLD) which is weaker than (US). They proved two versions of core equivalence theorems corresponding to (US) and (PLD) respectively. The condition (BRS) is maintained throughout their paper.

in Theorem 5.11 a core equivalence by utilizing a version of global substitutability ((CD), Assumption 5.10) proposed by Rustichini and Yannelis [56] without assuming local substitutability. Finally, all the proofs are given in Section 6.

2. Mathematical preliminaries

2.1. **Some Measure Space Theories.** For a topological (or Banach) space Y, we denote the Borel σ-algebra of Y which is defined as the σ-algebra generated by open subsets of Y by $\mathscr{B}(Y)$ and the set of Borel probability measures by $\mathcal{M}(Y)$. We begin with the next well-known fact ([49, Theorem I.2.1]).

Fact 2.1. Let X be a separable metric space. Then, for every measurable set B of measure space $(X, \mathcal{B}(X), \lambda)$, there exists a closed set $C_B \subset B$ satisfying $\lambda(C_B) = \lambda(B)$ and for any closed set $C \subset B$ with $\lambda(C) = \lambda(B)$, $C_B \subset C$.

The closed set C_B in Fact 2.1 is called the *support* of B and denoted by $\mathit{support}(B)$. Then, the next fact is also well known ([49, Theorem I.3.9]).

Fact 2.2. Let Y_1 and Y_2 be a complete separable metric space and $B_1 \in \mathscr{B}(Y_1)$ and $B_2 \subset Y_2$. Let $f : B_1 \to Y_2$ be a measurable and injective (one-to-one) map with $f(B_1) = B_2$. Then, $B_2 \in \mathscr{B}(Y_2)$.

Let a measurable space $(A, \mathcal{A})$ and a set E be given. For a map $f : E \to A$, define $\sigma(f) = \{f^{-1}(B)|\ B \in \mathcal{A}\}$; the smallest σ-algebra on E that makes f measurable. One has ([1, Theorem 4.41])

Fact 2.3. Let $f : E \to A$ and $g : E \to Y$, where Y is complete and separable[7]. Then, g is $\sigma(f)$-measurable if and only if there exists an $\mathcal{A}$-measurable map $h : A \to Y$ such that $g = h \circ f$.

Given any probability space $(A, \mathcal{A}, \lambda)$, a function on A is called *almost one-to-one* if it is one-to-one on A except for some λ-null set of $\mathcal{A}$. We have ([7, Theorem 9.6.3]),

Fact 2.4. Let Y be a complete separable metric space and $\boldsymbol{m} \in ca(Y)$. If $(Y, \mathscr{B}(Y), \boldsymbol{m})$ is an atomless measure space, there is an almost one-to-one map from $([0,1], \mathscr{B}([0,1]), dx)$ to Y, where dx is the Lebesgue measure on the Borel σ-algebra $\mathscr{B}([0,1])$[8].

Let f be a Borel measurable map from $(A, \mathcal{A}, \lambda)$ to a topological space X. The direct image measure $\lambda \circ f^{-1}$ is denoted by $f_*\lambda$. The operator $_*\lambda$ is a map from the set of all measurable maps of A to X that is denoted by $L^0(A, X)$ to $\mathcal{M}(X)$. We then have ([22, Lemma 2.1])

Fact 2.5. For an atomless measure space $(A, \mathcal{A}, \lambda)$, the map $_*\lambda : L^0(A, X) \to \mathcal{M}(X)$ is surjective.

[7] We can replace the range space $\mathbb{R}$ in the cited result by the Polish space X; see [25, footnote 26].

[8] In this notation, the Lebesgue integral is understood to be denoted as $\int f(x)dx$ rather than $\int f(x)d(dx)$.

A *measure algebra* is a pair $(\mathcal{A}, \lambda)$ where $\mathcal{A}$ is a *Boolean σ-algebra* with binary operations $\wedge$ and $\vee$, a unary operation ^{c} and λ is a real valued function satisfying the following conditions:(i) $\lambda(B) = 0$ if and only if $B = \emptyset$, where $\emptyset = A^c$ and $A = \emptyset^c$ are the smallest and the largest elements in $\mathcal{A}$, respectively; (ii) $\lambda(\vee_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n)$ for every sequence $\{E_n\}$ in $\mathcal{A}$ with $E_n \cap E_m = \emptyset$ whenever $m \neq n$. A map $\Phi : \mathcal{A} \to \mathscr{B}$ between measure algebras $(\mathcal{A}, \lambda)$ and $(\mathscr{B}, \mu)$ is called *homomorphism* if it is one-to-one, $\Phi(A^c) = \Phi(A)^c$, $\Phi(A \vee B) = \Phi(A) \vee \Phi(B)$ and $\lambda(A) = \mu(\Phi(A))$. Measure algebras $(\mathcal{A}, \lambda)$ and $(\mathscr{B}, \mu)$ are *isomorphic* if there exists a homomorphism which is onto.

A *subalgebra* of $\mathcal{A}$ is a subset of $\mathcal{A}$, which contains A and is closed under the Boolean operation $\wedge$, $\vee$, and ^{c}. The *order* $\leq$ on $\mathcal{A}$ is given by $B \leq C$ if and only if $B = B \wedge C$. A subalgebra $\mathcal{U}$ of $\mathcal{A}$ is *order-closed* with respect to $\leq$ if any nonempty upwards directed subsets of $\mathcal{U}$ with their supremum in $\mathcal{A}$ have the supremum in $\mathcal{U}$. A subset $\mathcal{U} \subset \mathcal{A}$ *completely generates* $\mathcal{A}$ if the smallest order-closed subalgebra in $\mathcal{A}$ containing $\mathcal{U}$ is $\mathcal{A}$ itself. The *Maharam type* of $(\mathcal{A}, \lambda)$ is the smallest cardinal of any subset $\mathcal{U}$, which completely generates $\mathcal{A}$.

Let $(A, \mathcal{A}, \lambda)$ be a finite measure space. We define an equivalence relation on $\mathcal{A}$ by $E \sim F$ if and only if $\lambda(E \Delta F) = 0$, where $E \Delta F = (E \wedge F^c) \vee (E^c \wedge F)$. The quotient space is denoted by $\hat{A} = A/\sim$. The equivalence class represented by $E \in \mathcal{A}$ is denoted $\hat{E}$. Then, the lattice operation and the unary operation ^{c} is defined naturally on $\hat{A}$, $\hat{E} \vee \hat{F} = \widehat{E \cup F}$ (union), $\hat{E} \wedge \hat{F} = \widehat{E \cap F}$ (intersection), $\hat{E}^c = \widehat{A \backslash E}$ (complement). The pair $(\hat{A}, \hat{\lambda})$ is a measure algebra associated with $(A, \mathcal{A}, \lambda)$, where $\hat{\lambda}(\hat{E}) = \lambda(E)$. Moreover, $(\hat{A}, \hat{\lambda})$ becomes a complete metric space by the metric $\rho(E, F) = \lambda(E \Delta F)$ (see [1], Lemma 13.13). The measure algebra $(\hat{A}, \hat{\lambda})$ is *separable* if it is a separable metric space. The Maharam type of $(A, \mathcal{A}, \lambda)$ is defined to be that of $(\hat{A}, \hat{\lambda})$.

Let $\mathcal{A}_E = \{A \cap E |\ A \in \mathcal{A}\}$ the sub-σ algebra of $\mathcal{A}$ restricted to $E \in \mathcal{A}$. We denote the restriction of λ to $\mathcal{A}_E$ by λ_E, or $\lambda_E(B) = \lambda(B)$ for every $B \in \mathcal{A}_E$. A finite measure space $(A, \mathcal{A}, \lambda)$ is (Maharam type) *homogeneous* if for every $E \in \mathcal{A}$ with $\lambda(E) > 0$, the Maharam type of $(E, \mathcal{A}_E, \lambda_E)$ is equal to $(A, \mathcal{A}, \lambda)$. It is easy to see (e.g., [52, p.838])

Fact 2.6. A finite measure space $(A, \mathcal{A}, \lambda)$ is atomless if and only if for every $E \in \mathcal{A}$ with $\lambda(E) > 0$, the Maharam type of $(E, \mathcal{A}_E, \lambda_E)$ is infinite.

This fact motivates the next definition.

Definition 2.7. A finite measure space $(A, \mathcal{A}, \lambda)$ is *saturated* (or *super-atomless*) if for every $E \in \mathcal{A}$ with $\lambda(E) > 0$, the Maharam type of $(E, \mathcal{A}_E, \lambda_E)$ is uncountable.

Lebesgue space is homogeneous, and its Maharam type is countable ([13, 331X, p.130]). Note that the saturated measure spaces are not necessarily Maharam homogeneous. Typical examples of the (homogeneous) saturated measure spaces are the atomless *Loeb space* [36], product spaces of the form $[0,1]^{\mathbf{c}}$ and $\{0,1\}^{\mathbf{c}}$, where $\mathbf{c}$ is an uncountable cardinal, $[0,1]$ equipped with the Lebesgue measure, and $\{0,1\}$ the "half-half" measure. Measure algebras of $[0,1]^{\mathbf{c}}$ and $\{0,1\}^{\mathbf{c}}$ are homogeneous with their Maharam type $\mathbf{c}$ and are isomorphic whenever $\mathbf{c}$ is infinite cardinal (see [13, Theorems 331I and 331K]), and they are separable if and only if $\mathbf{c}$ is countable.

Let $(A, \mathcal{A}, \lambda)$ be a finite measure space, X and Y be complete separable metric spaces. The next definition reveals a crucial property of the saturated spaces.

Definition 2.8. A finite measure space $(A, \mathcal{A}, \lambda)$ is said to satisfy the *saturation property* for a measure $\mu \in \mathcal{M}(X \times Y)$ if for every measurable function f on A with $f_*\lambda = \mu_X$, there exists a measurable function g on Y that satisfies $(f, g)_*\lambda = \mu$, where μ_X means the marginal distribution of μ on X.

Let $L^1(\lambda)$ be the set of all λ-integrable functions on A,

$$L^1(\lambda) = \left\{ f : A \to \mathbb{R} \middle| \int_A |f(a)| d\lambda < +\infty \right\}.$$

Denote by $L^1_E(\lambda)$ be the vector subspace of $L^1(\lambda)$ whose element is a restriction of each function in $L^1(\lambda)$ to E. It is well known that $(\hat{A}, \hat{\lambda})$ is separable if and only if $L^1(\lambda)$ is separable ([1, Lemma 13.14]). The next characterization of the saturation is well known (see [13, 22]) and would be sometimes useful.

Fact 2.9. Let $(A, \mathcal{A}, \lambda)$ be a finite measure space and X and Y be complete separable metric spaces. Then, the following conditions are equivalent.
(a) $(A, \mathcal{A}, \lambda)$ is saturated,
(b) $(A, \mathcal{A}, \lambda)$ is atomless and satisfies the saturation property for every $\mu \in \mathcal{M}(X \times Y)$,
(c) $L^1_E(\lambda)$ is nonseparable for every $E \in \mathcal{A}$ with $\lambda(E) > 0$.

Let $([0, 1], \mathcal{L}([0, 1]), dx)$ be the usual *Lebesgue space*, i.e., the completion of the measure space $([0, 1], \mathcal{B}([0, 1]), dx)$. Keisler–Sun [22] proved

Fact 2.10. Let $(A, \mathcal{A}, \lambda)$ be a finite measure space, X and Y be complete separable metric spaces and $f : A \to X$, $g : A \to Y$ be measurable maps, and assume that $f_*\lambda$ is atomless. Suppose that $(A, \mathcal{A}, \lambda)$ has the saturation property for $(f, g)_*\lambda$, but the Lebesgue space $([0, 1], \mathcal{L}([0, 1]), dx)$ does not. Then, $(A, \mathcal{A}, \lambda)$ is saturated.

The next result is known as *Egorov's Theorem* ([11, p. 97]).

Fact 2.11. Let (X, d_X) be a metric space and $(A, \mathcal{A}, \lambda)$ be a finite measure space. If (f_n) is a sequence of measurable maps on A to X that converges a.e to a map $f : A \to X$, then:
(i) f is measurable,
(ii) for every measurable set $E \in \mathcal{A}$ and every $\epsilon > 0$, there exists a set $F \in \mathcal{A}$ with $F \subset E$ and $\lambda(E \backslash F) < \epsilon$ such that (f_n) converges uniformly f on F.

Let $\{K^n\}$ be an increasing sequence of closed subsets of a compact metric space K converging to K in the topology of closed convergence. If $\boldsymbol{q}_n : K^n \to \mathbb{R}$ $t \mapsto q_n(t)$ is continuous, we write $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$ if $\boldsymbol{q} = q(t) \in C(K)$ and for every subsequence n_k and $t^{n_k} \in K^{n_k}$ with $t^{n_k} \to t$, $q_{n_k}(t^{n_k}) \to q(t)$. We have the following:

Fact 2.12. Let $\{\boldsymbol{m}_n\}$ be a bounded sequence in $ca(K)$ with *support*$(\boldsymbol{m}_n) \subset K^n$ and $\boldsymbol{m}_n \to \boldsymbol{m}$ in the weak* topology, and $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$. Then, $\boldsymbol{q}_n\boldsymbol{m}_n \to \boldsymbol{q}\boldsymbol{m}$ (Mas-Colell [41]).

Let $(K^n, \boldsymbol{q}_n)$ be a sequence as above. We say that it is equicontinuous if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $t, s \in K^n$ with $d(t, s) \le \delta$, $|\boldsymbol{q}_n(t) - \boldsymbol{q}_n(s)| \le \epsilon$. Mas-Colell [41] also proved the following:

Fact 2.13. Let $\{K^n\}$ be a sequence of closed sets of a compact metric space K with $K^n \subset K^{n+1} \subset \cdots \to K$ in the topology of closed convergence and $\{\boldsymbol{q}_n\}$ a sequence in $C(K)$ with $\|\boldsymbol{q}_n\| \leq 1$. If $(K^n, \boldsymbol{q}_n)$ is equicontinuous, then there is a subsequence n_k and $\boldsymbol{q} \in C(K)$ with $(K^{n_k}, \boldsymbol{q}_{n_k}) \to (K, \boldsymbol{q})$.

2.2. **Topological Vector Lattices.** Let L and M be Banach spaces. The bilinear form for the duality pair $\langle L, M\rangle$ is denoted by $\langle \boldsymbol{z}, \boldsymbol{r}\rangle$ or $\boldsymbol{zr}$ for short, where $\boldsymbol{z} \in L$ and $\boldsymbol{r} \in L^*$. However, $\boldsymbol{rz}$ denoted by the reverse order is more convenient for economic applications, and we follow this notation in the following. We summarize the basic properties of the spaces used in the text.

The space of all bounded sequences

$$\ell^\infty = \left\{\boldsymbol{x} = (x^t) |\ sup_{t\geq 1}|x^t| < +\infty\right\},$$

is a nonseparable Banach space with respect to the norm $\|\boldsymbol{x}\| = sup_{t\geq 1}|x^t|$ for $\boldsymbol{x} \in \ell^\infty$ with the dual space

$$ba = \left\{\boldsymbol{q} : 2^{\mathbb{N}} \to \mathbb{R} \middle|\ sup_{E\subset\mathbb{N}}|\boldsymbol{q}(E)| < +\infty, \boldsymbol{q}(E \cup F) = \boldsymbol{q}(E) + \boldsymbol{q}(F) \right.$$
$$\left. \text{whenever } E \cap F = \emptyset \right\}$$

which is the space of bounded and finitely additive set functions on $\mathbb{N}$ under the duality $\boldsymbol{qz} = \int \boldsymbol{z}d\boldsymbol{q}$ with $\boldsymbol{z} \in \ell^\infty$ and $\boldsymbol{q} \in ba$. Let $\ell^\infty_+ = \{\boldsymbol{z} \in \ell^\infty |\ \boldsymbol{z} \geq \boldsymbol{0}\}$ be the nonnegative orthant of ℓ^∞. The topology on ℓ^∞ defined by the point-wise convergence on ba is called weak topology and denoted as $\sigma(\ell^\infty, ba)$.

It is well known that the dual space of the space of all summable sequences,

$$\ell^1 = \left\{\boldsymbol{p} = (p^t) \middle|\ \sum_{t=1}^{\infty} |p^t| < +\infty\right\},$$

is ℓ^∞, and ℓ^1 is a separable Banach space with the norm $\|\boldsymbol{p}\| = \sum_{t=1}^{\infty} |p^t|$. The nonnegative orthant ℓ^1_+ is defined similarly as ℓ^∞_+. Thus, the space ℓ^1 is isomorphic to the subspace ca of ba,

$$ca = \left\{\boldsymbol{p} \in ba \middle|\ \boldsymbol{p}(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \boldsymbol{p}(E_n) \text{ whenever } E_i \cap E_j = \emptyset\ (i \neq j)\right\},$$

which is the space of the bounded and countably additive set functions on $\mathbb{N}$. The topology on ℓ^∞ defined by the point-wise convergence on ℓ^1 is called weak* topology and denoted as $\sigma(\ell^\infty, \ell^1)$.

The set function $\boldsymbol{q} \in ba$ is called *purely finitely additive* if $\boldsymbol{q}' = 0$ whenever $\boldsymbol{q}' \in ca$ and $\boldsymbol{0} \leq \boldsymbol{q}' \leq \boldsymbol{q}$. The relation between the ba and ca is made clear by the next fundamental theorem,

Fact 2.14. If $\boldsymbol{q} \in ba$ and $\boldsymbol{q} \geq 0$, then there exist set functions $\boldsymbol{q}_c \geq 0$ and $\boldsymbol{q}_p \geq 0$ in ba such that $\boldsymbol{q}_c$ is countably additive and $\boldsymbol{q}_p$ is purely finitely additive and satisfy $\boldsymbol{q} = \boldsymbol{q}_c + \boldsymbol{q}_p$. This decomposition is unique ([68]).

Let (K, d_K) be a compact metric space. The space $ca(K)$ is the set of bounded countably additive set functions (signed measures) on K,

$$ca(K) = \left\{ \boldsymbol{m} : \mathscr{B}(K) \to \mathbb{R} \middle| \; sup_{E \subset K} |\boldsymbol{m}(E)| < +\infty, \; \boldsymbol{m}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \boldsymbol{m}(E_i) \right.$$
$$\left. \text{whenever } E_i \cap E_j = \emptyset \; (i \neq j) \right\}.$$

Then, $ca(K)$ is a Banach space with respect to the norm

$$\|\boldsymbol{m}\| = sup \left\{ \sum_{i=1}^{n} |\boldsymbol{m}(E_i)| \middle| \; E_i \cap E_j = \emptyset \text{ for } i \neq j, n \in \mathbb{N} \right\} \text{ for } \boldsymbol{m} \in ca(K).$$

Let $C(K)$ be the set of all continuous functions on K. $C(K)$ is also a separable Banach space with respect to the norm $\|\boldsymbol{q}(t)\| = sup\{|q(t)|| \; t \in K\}$ for $\boldsymbol{q} = q(t) \in C(K)$. The *Riesz representation theorem* ([53, p. 357]) asserts that the dual space of $C(K)$ is $ca(K)$, or $C(K)^* = ca(K)$ under the duality $\langle \boldsymbol{m}, \boldsymbol{q} \rangle = \int q(t) d\boldsymbol{m}(t)$ with $\boldsymbol{m} \in ca(K)$ and $\boldsymbol{q} = q(t) \in C(K)$ (see [1, Theorem 14.15]). The topology on $ca(K)$ defined by the point-wise convergence on $C(K)$ is called the weak* topology and denoted by $\sigma(ca(K), C(K))$. Bounded subsets of ℓ^∞ and $ca(K)$ are $\sigma(\ell^\infty, \ell^1)$ and $\sigma(ca(K), C(K))$-weakly compact, respectively, i.e., that the weak* closure of the sets are weak* compact by *Banach–Alaoglu's theorem.*

Fact 2.15. If L is a Banach space, then the unit ball of L^*, $B = \{\boldsymbol{r} \in L^* | \; \|\boldsymbol{r}\| \leq 1\}$ is compact in the $\sigma(L^*, L)$-topology. Moreover, if L is a separable Banach space, then norm-bounded subset of L^* is a compact metric space ([54, pp. 68-70]).

We use the following notations for the vector space orderings. For $\boldsymbol{x} = (x^t) \in \mathbb{R}^k$ or ℓ^∞ or ℓ^1, $\boldsymbol{x} \geq \boldsymbol{0}$ means that $x^t \geq 0$ for all t and $\boldsymbol{x} > 0$ means that $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{x} \neq \boldsymbol{0}$. $\boldsymbol{x} \gg \boldsymbol{0}$ means that $x^t > 0$ for all t. Finally, for $\boldsymbol{x} = (x^t) \in \ell^\infty$, we denote by $\boldsymbol{x} \ggg \boldsymbol{0}$ if and only if there exists an $\epsilon > 0$ such that $x^t \geq \epsilon$ for all t. For $\boldsymbol{m} \in ca(K)$, $\boldsymbol{m} \geq \boldsymbol{0}$ means that $\boldsymbol{m}(B) \geq 0$ for every $B \in \mathscr{B}(K)$. $\boldsymbol{m} > \boldsymbol{0}$ means that $\boldsymbol{m} \geq \boldsymbol{0}$ and $\boldsymbol{m} \neq \boldsymbol{0}$. The nonnegative orthant of $ca(K)$ or $ca_+(K) = \{\boldsymbol{m} \in ca(K) | \; \boldsymbol{m} \geq \boldsymbol{0}\}$ is nothing but the set of Borel measures $\mathcal{M}(K)$ on K. Because we have assumed K to be a compact metric space, $\mathcal{M}(K)$ is a complete and separable metric space in the weak* topology; see[66].

For $t \in K$, the Dirac measure δ_t is defined by $\delta_t(E) = 1$ when $t \in E$, $\delta_t(E) = 0$ when $t \notin E$. Because (K, d) is a compact metric space, it is separable. Hence, there exists a countable dense subset $\{t_1, t_2 \dots\}$ of K. Let $LS(t_1 \dots t_n)$ be the linear space spanned by $\{\delta_{t_1} \dots \delta_{t_n}\}$. It is well known that the set $\cup_{n=1}^{\infty} LS(t_1 \dots t_n)$ is dense in $ca(K)$ in the weak* topology.

These spaces are examples of *locally convex topological vector spaces*, that is, real vector spaces equipped with a locally convex Hausdorff topology (i.e., topology having a neighborhood base at $\boldsymbol{0}$ consisting of convex sets) such that the addition and the scalar multiplication are jointly continuous. Moreover, they are also topological vector lattices. In general, an *ordered vector space* is a vector space L endowed with a reflexive, transitive and antisymmetric relation $\leq$ that satisfies (i) if $\boldsymbol{x} \leq \boldsymbol{y}$ and $a \in \mathbb{R}_+$, then $a\boldsymbol{x} \leq a\boldsymbol{y}$, (ii) if $\boldsymbol{x} \leq \boldsymbol{y}$, then $\boldsymbol{x} + \boldsymbol{z} \leq \boldsymbol{y} + \boldsymbol{z}$ for each $\boldsymbol{z} \in L$. Let L be an ordered vector space and $S \subset L$. An element $\boldsymbol{y} \in L$ is an *upper bound* of S if and only if $\boldsymbol{x} \le \boldsymbol{y}$ for every $\boldsymbol{x} \in S$. An element $\sup S$ is called the *supremum* (*least upper bound*) if it is an upper bound of S and $\sup S \le \boldsymbol{y}$ for every upper bound $\boldsymbol{y}$ of S. Similarly, $\boldsymbol{z}$ is a *lower bound* of S if $\boldsymbol{z} \le \boldsymbol{x}$ for every $\boldsymbol{x} \in S$. $\inf S$ is the *infimum* (*greatest lower bound*) if it is a lower bound of S and $\boldsymbol{z} \le \inf S$ for every lower bound $\boldsymbol{z}$ of S. We usually write $\boldsymbol{x} \vee \boldsymbol{y}$ rather than $\sup\{\boldsymbol{x}, \boldsymbol{y}\}$ and $\boldsymbol{x} \wedge \boldsymbol{y}$ rather than $\inf\{\boldsymbol{x}, \boldsymbol{y}\}$. If every pair $\boldsymbol{x}, \boldsymbol{y}$ of an ordered vector space L has the supremum $\boldsymbol{x} \vee \boldsymbol{y}$ and the infimum $\boldsymbol{x} \wedge \boldsymbol{y}$, then we call L a *vector lattice* or *Riesz space*. We write $\boldsymbol{x}_+ = \boldsymbol{x} \vee \mathbf{0}$ and $\boldsymbol{x}_- = (-\boldsymbol{x}) \vee \mathbf{0}$ and call the *positive part* and the *negative part* of $\boldsymbol{x}$, respectively. Then, $\boldsymbol{x} = \boldsymbol{x}_+ - \boldsymbol{x}_-$, and we write $|\boldsymbol{x}| = \boldsymbol{x}_+ + \boldsymbol{x}_-$ and call the *absolute value* of $\boldsymbol{x}$. A topological vector lattice L is called a *Banach lattice* if it is a Banach space and satisfies $\|\boldsymbol{x}\| \le \|\boldsymbol{y}\|$ whenever $|\boldsymbol{x}| \le |\boldsymbol{y}|$. The following facts together with Fact 2.15 are also included as the "essential mathematical structures ([44, p. 1838])".

Fact 2.16. Let L be a locally convex topological vector space and L' its topological dual. If A and B are disjoint convex sets such that one of which has an interior point, then there is a nonzero continuous linear functional $\boldsymbol{r} \in L'$ such that $\boldsymbol{rx} \le \boldsymbol{ry}$ for each $\boldsymbol{x} \in A$ and each $\boldsymbol{y} \in B$ (*Hahn–Banach separation theorem*, [1, Theorem 5.67]).

Fact 2.17. Let L be a vector lattice and let $\boldsymbol{x}_1 \dots \boldsymbol{x}_n, \boldsymbol{z} \in L_+$ satisfy $\boldsymbol{z} \le \sum_{i=1}^n \boldsymbol{x}_i$. Then, there are $\boldsymbol{z}_1 \dots \boldsymbol{z}_n \in L_+$ that satisfy $\sum_{i=1}^n \boldsymbol{z}_i = \boldsymbol{z}$ and $\boldsymbol{z}_i \le \boldsymbol{x}_i$ (Riesz decomposition theorem, [1, p. 319]).

2.3. **Integrations of Vector-Valued Maps.** Let $(A, \mathcal{A}, \lambda)$ be a finite measure space, L a Banach space and $M = L^*$ its norm dual. A map $f : A \to L$ is called *simple* if it is of the form $f(a) = \sum_{i=1}^n \boldsymbol{x}_i \mathbf{1}_{A_i}(a)$, where $\boldsymbol{x}_i \in L$, A_i are measurable partitions of A with $\cup_{i=1}^n A_i = A$ and $\mathbf{1}_{A_i}(a)$ is the characteristic function of A_i, or $\mathbf{1}_{A_i}(a) = 1$ when $a \in A_i$ and $\mathbf{1}_{A_i}(a) = 0$ when $a \notin A_i$. A map $f : A \to L$ is said to be *strongly measurable* if there exists a sequence (f_n) of simple functions with $\|f_n(a) - f(a)\| \to 0$ a.e. A strongly measurable function f is said to be *Bochner integrable* if $\int_A \|f_n(a) - f(a)\| d\lambda \to 0$. In this case, we denote $\int_A f(a) d\lambda = \lim_{n\to\infty} \int_A f_n(a) d\lambda$ and call $\int_A f(a) d\lambda$ the *Bochner integral*. It is well known ([10, Theorem 2, p. 45]) that a strongly measurable map f is Bochner integrable if and only if $\int_A \|f(a)\| d\lambda < \infty$.

A map $f : A \to L$ is said to be *weakly measurable* if for each $\boldsymbol{r} \in L^*$, $\boldsymbol{r}f(a)$ is measurable. A weakly measurable map $f(a)$ is said to be *Pettis integrable* if there exists an element $\xi_f \in L$ such that for each $\boldsymbol{r} \in L^*$, $\boldsymbol{r}\xi_f = \int_A \boldsymbol{r}f(a) d\lambda$. The vector ξ_f is denoted by $\int_A f(a) d\lambda$ and called *Pettis integral* of f.

A map $f : A \to L^*$ is said to be *weak* measurable* if for each $\boldsymbol{z} \in L$, $\boldsymbol{z}f(a)$ is measurable. A weak* measurable map $f(a)$ is said to be *Gelfand integrable* if there exists an element $\pi_f \in L^*$ such that for each $\boldsymbol{z} \in L$, $\boldsymbol{z}\pi_f = \int_A \boldsymbol{z}f(a) d\lambda$. The vector π_f is denoted by $\int_A f(a) d\lambda$ and called *Gelfand integral* of f.

In general, let L be a locally convex topological vector space and L' be its topological dual. A map $f : A \to L$ is said to be *weakly measurable* if for each $\boldsymbol{r} \in L'$, $\boldsymbol{r}f(a)$ is measurable. A weakly measurable map $f(a)$ is said to be *Pettis integrable* if there exists an element $\xi_f \in L$ such that for each $\boldsymbol{r} \in L'$, $\xi_f \boldsymbol{r} = \int_A \boldsymbol{r}f(a) d\lambda$. The vector ξ_f is denoted by $\int_A f(a)d\lambda$ and called *Pettis integral* of f. Those integral concepts can be generalized to correspondences.

Fact 2.18. Let L be a Banach space and L^* its norm dual space. If $f : A \to L^*$ is weak* measurable and $\boldsymbol{z}f(a)$ is integrable function for all $\boldsymbol{z} \in L$, then f is Gelfand integrable ([10, pp. 53–54]).

Fact 2.19. Let $\{f_n\}$ be a sequence of Gelfand integrable functions from A to L^* which converges a.e. to f in the weak* topology. Then, it follows that $\int_A f_n(a)d\lambda \to \int_A f(a)d\lambda$ in the weak* topology.

For instance, let $L = \ell^1$. Then $M = L^* = \ell^\infty$. A map $f : A \to \ell^\infty$ is weak* measurable if for each $\boldsymbol{p} \in \ell^1$, $\boldsymbol{p}f(a)$ is measurable. The Gelfand integrable of f is an element $\boldsymbol{x}_f \in \ell^\infty$ such that for each $\boldsymbol{p} \in \ell^1$, $\boldsymbol{p}\boldsymbol{x}_f = \int_A \boldsymbol{p}f(a)d\lambda$. Similarly, a map $f : A \to ca(K)$ is weak* measurable if for each $\boldsymbol{q} \in C(K)$, $\boldsymbol{q}f(a)$ is a measurable function on $(A, \mathcal{A}, \lambda)$, and it is Gelfand integrable if there exists an element $\int_A f(a)d\lambda \in ca(K)$ such that for each $\boldsymbol{q} \in C(K)$, $\boldsymbol{q}\int_A f(a)d\lambda = \int_A \boldsymbol{q}f(a)d\lambda$. In particular, for every Borel set $B \in \mathscr{B}(K)$, the value of the measure $\int_A f(a)d\lambda$ at B is defined by $\int_A f(a)d\lambda(B) \equiv \int_A f(a)(B)d\lambda$.

The integral of a correspondences can be defined as in the finite dimensional cases. For instance let $\phi : A \to L^*$ be a correspondence and $\mathscr{L}(\phi) = \{f : A \to L^* |\, f \text{ is Gelfand integrable, } f(a) \in \phi(a)\ a.e\}$ the set of Gelfand integrable selections of ϕ. We then define

$$\int_A \phi(a)d\lambda = \left\{ \int_A f(a)d\lambda \,\middle|\, f \in \mathscr{L}(\phi) \right\}.$$

Let X be a topological space and F_n a sequence of subsets of X. The topological limes superior $Ls(F_n)$ is defined by the following:

$x \in Ls(F_n)$ if and only if there exists a subsequence F_{n_k} with $x_{n_k} \in F_{n_k}$ for all k and $x_{n_k} \to x$ $(k \to \infty)$.

The next fact is also well known (Fatou's lemma in ℓ dimensions, [18, p. 69]).

Fact 2.20. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions of a measure space $(A, \mathcal{A}, \lambda)$ to $\mathbb{R}^\ell_+$. Suppose that $\lim_n \int_A f_n(a)d\lambda$ exists. Then, there exists an integrable function $f : (A, \mathcal{A}, \lambda) \to \mathbb{R}^\ell_+$ such that
(*a*) $f(a) \in Ls(f_n(a))$ a.e. in A,
(*b*) $\int_A f(a)d\lambda \le \lim_{n\to\infty} \int_A f_n(a)d\lambda$.

Consider a sequence of maps $\phi_n : A \to X$. For each $a \in A$, if $Ls(\phi_n(a)) \neq \emptyset$, we can define a correspondence $Ls(\phi_n(\cdot)) : A \to X, a \mapsto Ls(\phi_n(a))$. The next theorem is attributed to Khan–Sagara–Suzuki [30], which is an infinite dimensional version of the Fatou's lemma and plays a crucial role in our proofs (see also [14]).

Fact 2.21. Let $(A, \mathcal{A}, \lambda)$ be a complete and finite measure space that is saturated and L^* be the dual space of a separable Banach space L. Let $f_n : A \to L^*$ be a sequence of Gelfand integrable mappings from A to L^* such that there exists an integrable function $g(a)$ with $\sup_n \|f_n(a)\| \le g(a)$ a.e.

Then, there exists a Gelfand integrable map $f : A \to L^*$ with

$$\int_A f(a)d\lambda \in Ls\left(\int_A f_n(a)d\lambda\right) \text{ and } f(a) \in Ls(f_n(a))\ a.e.$$

Podczeck–Sun–Yannelis ([52, 57]) proved the following:

Fact 2.22. Let $(A, \mathcal{A}, \lambda)$ be a finite measure space and L^* be the dual space of a separable Banach space L. Then, the following conditions are equivalent.
(i) $\int_A \phi(a) d\lambda$ is convex for every correspondence $\phi : A \to L^*$,
(ii) the measure space $(A, \mathcal{A}, \lambda)$ is saturated.

Remark 2.23. Recall that a topological space is a Suslin space if and only if it is the image of a continuous map from a complete and separable metric space (Polish space). A separable Banach space L is a Suslin space, and its dual space L^* endowed with the weak* topology is also a Suslin space (see [64, p. 67]). Hence, ℓ^∞ with the weak* topology $\sigma(\ell^\infty, \ell^1)$ and $ca(K)$ with the weak* topology $\sigma(ca(K), C(K))$ for a compact metric space K are examples of Suslin spaces, because ℓ^1 and $C(K)$ are separable Banach spaces when K is compact ([12, p. 437]). A map from a measure space $(A, \mathcal{A}, \lambda)$ to a locally convex Suslin space is Borel measurable if and only if it is weakly measurable ([64, Theorem 1]). Because $\sigma(\ell^\infty, \ell^1)$ and $\sigma(ca(K), C(K))$ are locally convex, maps from $(A, \mathcal{A}, \lambda)$ to ℓ^∞ or $ca(K)$ are weak* measurable if and only if they are Borel measurable (with respect to the weak*topologies).

3. Basic model

3.1. **Setup and Assumptions.** Let L be a Banach lattice with its order relation denoted by $\geq$ that is a dual space of the separable Banach lattice M, or $L = M^*$. The order relation on M is also denoted by $\geq$ if there is no danger of confusion. As usual, we define $\xi > \mathbf{0}$ if and only if $\xi \geq \mathbf{0}$ and $\xi \neq \mathbf{0}$ and $L_+ = \{\xi \in L | \ \xi \geq \mathbf{0}\}$ ($L_- \equiv -L_+$), and similarly for M. Let Z be a convex and weak* closed subset of L_+. Throughout the paper, the commodity space is assumed to be $Q = \mathbb{R}^k \times L$, and the (common) consumption set is $X = \mathbb{R}^k_+ \times Z$. Typical elements of X are called the consumption vectors and denoted by $\xi = (x, \boldsymbol{x})$, $\zeta = (z, \boldsymbol{z})$, $\phi = (u, \boldsymbol{m})$, $\psi = (y, \boldsymbol{n})$ and so on, where $u, x, y, z \in \mathbb{R}^k_+$ and $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{x}, \boldsymbol{z} \in Z$. Accordingly, we assume the price space to be $P = \mathbb{R}^k \times M$, and its elements are called price vectors that are often denoted as $\pi = (p, \boldsymbol{p})$ or $\rho = (q, \boldsymbol{q})$, $p, q \in \mathbb{R}^k_+$ and $\boldsymbol{p}, \boldsymbol{q} \in M_+$.

As usual, a preference $\succsim \subset X \times X$ is a complete, transitive, and reflexive binary relation on X. We denote $(\xi, \zeta) \in \succsim$ by $\xi \succsim \zeta$. $\xi \prec \zeta$ means that $(\xi, \zeta) \notin \succsim$. Let $\mathcal{P}$ be the set of allowed preference relations. The assumptions for the preferences are as follows:

Assumption 3.1. (PR) (i) For every $\succsim \in \mathcal{P}$, $\succsim \subset X \times X$ is complete, transitive, and reflexive, which is closed in $X \times X$ in the weak* topology,
(ii) (Monotonicity). For every $\succsim \in \mathcal{P}$, $\xi \prec \zeta$ whenever $\xi, \zeta \in X$ and $\xi < \zeta$.

As stated in Section 1, we do not need the convexity for the preferences, compared with [6, 35, 40, 46, 47, 48, 61]. Because $\mathcal{P} \subset \mathcal{F}(X \times X)$ by (i), we can endow $\mathcal{P}$ with the topology of closed convergence on $\mathcal{F}(X \times X)$ ([18, pp. 15–19]).

An endowment vector is an element of X. We denote the set of all endowment vectors by Ω and assume it to be weak* compact subset of X. Throughout the paper, we consider the measurable space of consumers' characteristics $(\mathcal{P} \times \Omega, \mathscr{B}(\mathcal{P} \times \Omega))$.

3.2. **Equilibria and Core.** Let $(A, \mathcal{A}, \lambda)$ be a complete probability space of the consumers. The definition of the economy is standard.

Definition 3.2. A *LIE* $\mathcal{E}$ is a Borel measurable mapping $\mathcal{E} : A \to \mathcal{P} \times \Omega$ defined by $a \mapsto (\succsim_a, \omega(a))$. The economy is called *saturated* (or *super-atomless*) if the measure space $(A, \mathcal{A}, \lambda)$ is saturated.

An *allocation* is a Gelfand integrable map $\xi : A \to X$ a.e. An allocation is said to be *feasible* if $\int_A \xi(a) d\nu \leq \int_A \omega(a) d\nu$. It is called *exactly feasible* if $\int_A \xi(a) d\nu = \int_A \omega(a) d\nu$.

We now state a definition of a competitive equilibrium.

Definition 3.3. A pair (π, ξ) of a price vector $\pi \in P_+$ with $\pi \neq \mathbf{0}$ and an allocation $\xi : A \to X$ is called an *allocative equilibrium* of the economy $\mathcal{E}$ if the following conditions hold:
(E-1) $\pi\xi(a) \leq \pi\omega(a)$ and $\xi(a) \succsim_a \zeta$ whenever $\pi\zeta \leq \pi\omega(a)$ a.e.,
(E-2) $\int_A \xi(a) d\lambda = \int_A \omega(a) d\lambda$.

A nonnull measurable subset S of the measure space of consumers $(A, \mathcal{A}, \lambda)$ is called a coalition. We furnish the conventional definition of the core.

Definition 3.4. A coalition $S \subset A$ is said to *block* an allocation ξ if there exists an allocation ζ such that
(i) $\int_S \zeta(a) d\lambda \leq \int_S \omega(a) d\lambda$;
(ii) $\xi(a) \prec_a \zeta(a)$ on S.
A feasible allocation $\xi : A \to X$ belongs to the *core* of an economy $\mathcal{E}$ if there exist no coalitions $S \subset A$ which block ξ.

4. Realization of distributive equilibria

4.1. **LIE Representations for LDEs.** In this section, we assume that X is a complete and separable metric space (Polish space) in the weak* topology and that $\mathcal{P}$ is also a Polish space in the closed convergence topology ([18, p. 19]). These assumptions on X and $\mathcal{P}$ are satisfied for the models in Section 5. The distributive economy is defined as follows.

Definition 4.1. A *LDE* is a probability measure μ on the measurable space $(\mathcal{P} \times \Omega, \mathscr{B}(\mathcal{P} \times \Omega))$. A distributionalized economy μ is called *atomless* if it is an atomless probability measure.

A probability measure ν on $X \times \mathcal{P} \times \Omega$ is called an *allocation distribution* if $\nu_{\mathcal{P} \times \Omega} = \mu$. An allocation distribution is called *(exactly) feasible* if $\int_X \iota(\xi) d\nu_X = \int_\Omega \iota(\omega) d\mu_\Omega$, where ι is the inclusion map. Because $\iota(\xi) = \xi$ for all ξ, we hereafter denote $\int_X \iota(\xi) d\nu_X = \int_X \xi d\nu_X$, and so on. The distributive equilibrium is defined as follows.

Definition 4.2. A pair (π, ν) of a price vector $\pi \in P_+$ with $\pi \neq \mathbf{0}$ and an allocation distribution ν on $X \times \mathcal{P} \times \Omega$ is called a *distributive equilibrium* of the economy μ if the following conditions hold,
(D-1) $\nu(\{(\xi, \succsim, \omega) \in X \times \mathcal{P} \times \Omega | \ \pi\xi \leq \pi\omega$ and $\xi \succsim \eta$ whenever $\pi\eta \leq \pi\omega\}) = 1$,
(D-2) $\int_X \xi d\nu_X = \int_\Omega \omega d\mu_\Omega$,

(D-3) $\nu_{\mathcal{P}\times\Omega} = \mu$,

where the marginals of μ are denoted by subscripts, e.g., $\mu_{\mathcal{P}}$ denotes the marginal on $\mathcal{P}$, and so on.

Let $(A, \mathcal{A}, \lambda)$ be an atomless probability measure space for a measurable map $f : A \to \mathcal{P}\times\Omega$.

Definition 4.3. For an economy μ, a measurable map $\mathcal{E} : A \to \mathcal{P}\times\Omega$ such that $\mu = \mathcal{E}_*\lambda$ is called a *representation* of μ. The representation is called *saturated* if the measure space $(A, \mathcal{A}, \lambda)$ is saturated.

Note that a representation is not unique even if it exists. Because $\mathcal{P} \times \Omega$ is a Polish space, the representations of μ exists by Fact 2.5 in Section 2.1. Moreover, because the saturated measure spaces are atomless, the saturated representations also exist. Similarly, for every allocation distribution ν, a measurable map $(\xi, \mathcal{E}) : A \to X \times \mathcal{P}\times\Omega$, which satisfies $\nu = (\xi, \mathcal{E})_*\lambda$ is the representation of ν. The map $\xi : A \to X$ is only simply an allocation. The representations for ν also exist by the same reason for μ. We may call the existence of the representations for ν the *weak equivalence* of the individual and distributive equilibria ([25]).

A fundamental problem is the realization of the distributive equilibrium or the *strong equivalence* of the two equilibria; given an equilibrium ν of an economy μ and an individual economy $\mathcal{E}$, which represents μ, can we obtain an allocation ξ such that $(\xi, \mathcal{E})$ represents ν? The answer is generally negative for atomless measure spaces of the consumers. However, the positive answer is the rule for the saturated measure spaces.

Theorem 4.4. *Let $(A, \mathcal{A}, \lambda)$ be a saturated probability space. Let distributive economy μ and its equilibrium ν be given. For every individual economy $\mathcal{E} : A \to \mathcal{P}\times\Omega$ which represents μ, there exists an equilibrium allocation $\xi : A \to X$ such that $\nu = (\xi, \mathcal{E})_*\lambda$.*

4.2. **Symmetric Equilibria.** The concept of the symmetric equilibria in the next definition is attributed to [42] (see also [32]).

Definition 4.5. The equilibrium ν is called *symmetric* if there exists a measurable map $\theta : \mathcal{P} \times \Omega \to X$ such that $\nu(Graph(\theta)) = 1$, where $Graph(\theta) = \{(\xi, \succsim, \omega) \in X \times \mathcal{P} \times \Omega |\ \xi = \theta(\succsim, \omega)\}$.

If an equilibrium is symmetric, then the consumers with the identical characteristics consume the identical consumption vector.

Definition 4.6. Let a distributionalized economy μ and its equilibrium ν be given. A probability space $(A, \mathcal{A}, \lambda)$ *realizes* ν, or $(A, \mathcal{A}, \lambda)$ is a *realization* of ν, if every individual economy $\mathcal{E} : A \to \mathcal{P}\times\Omega$ which represents μ has a measurable map $\xi : A \to X$ such that $\nu = (\xi, \mathcal{E})_*\lambda$.

Let $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathcal{P} \times \Omega$ be an individual representation that is defined on the measure space of consumers of the distributionalized economy μ and define $\sigma(\mathcal{E}) = \{\mathcal{E}^{-1}(B) |\ B \in \mathscr{B}(\mathcal{P} \times \Omega)\}$; the smallest σ-algebra on A which makes $\mathcal{E}$ measurable. The following results are already known for the large atomless games [25].

Theorem 4.7. *Assume that X is a complete separable metric space and let (π, ν) be a distributive equilibrium of μ and $\mathcal{E}$ a representation of μ defined on $(A, \mathcal{A}, \lambda)$. Then, ν is symmetric if and only if $\nu = (\xi, \mathcal{E})_* \lambda$ for a $\sigma(\mathcal{E})$-measurable equilibrium allocation ξ of $\mathcal{E}$.*

Given any probability space $(A, \mathcal{A}, \lambda)$, recall that a function on A is called almost one-to-one if it is one-to-one on A except for some λ-null set of $\mathcal{A}$. If λ is atomless, we can show by Fact 2.4 in Section 2.1 that there exists an almost one-to-one Lebesgue representation, i.e., a representation defined on the Lebesgue space $([0,1], \mathscr{B}([0,1]), dx)$.

Theorem 4.8. *Let $\mathcal{E}$ be a representation of μ defined on $([0,1], \mathscr{B}([0,1]), dx)$. Assume that $\mathcal{E}$ is almost one-to-one.*
(a) If $\xi : [0,1] \to X$ is a competitive equilibrium allocation of $\mathcal{E}$ that is measurable in $\mathscr{B}([0,1])$, then $\nu = (\xi, \mathcal{E})_* \lambda$ is a symmetric equilibrium distribution of μ.
(b) Let $\xi : [0,1] \to X$ be any Gelfand integrable function that is measurable in $\mathscr{B}([0,1])$, and let $\nu = (\xi, \mathcal{E})_* dx$. If ν is an equilibrium distribution of μ, then ξ is an equilibrium allocation of $\mathcal{E}$ and ν is symmetric.

Theorem 4.9. *Let an atomless distributionalized economy μ and its equilibrium ν be given. Then, the following conditions are equivalent.*
(a) ν is symmetric,
(b) every atomless probability space is a realization of ν,
(c) every atomless nonsaturated probability space is a realization of ν,
(d) the measure space $([0,1], \mathscr{B}([0,1]), dx)$ is a realization of ν.

Corollary 4.10. *An atomless probability space $(A, \mathcal{A}, \lambda)$ realizes a nonsymmetric equilibrium of an atomless distributionalized economy μ if and only if it is saturated.*

5. Existence and core equivalence of allocative equilibria

5.1. **A Market with Infinite Time Horizon.** In this section, we set $L = \ell^\infty$ and $M = \ell^1$; then, the commodity space is $Q = \mathbb{R}^k \times \ell^\infty$. We assume $k \geq 1$. The commodity vectors are written by $\xi = (x, \boldsymbol{x})$, $\zeta = (z, \boldsymbol{z})$, where $x = (x^i), z = (z^i) \in \mathbb{R}^k$ and $\boldsymbol{x} = (\boldsymbol{x}^t), \boldsymbol{z} = (\boldsymbol{z}^t) \in \ell^\infty$.

The price vector is assumed to be a vector $\pi = (p, \boldsymbol{p}) \in P_+ = \mathbb{R}^k_+ \times \ell^1_+$, where $p = (p^i) \in \mathbb{R}^k_+$ and $\boldsymbol{p} = (\boldsymbol{p}^t) \in \ell^1_+$. The value of a commodity $\xi = (x, \boldsymbol{x}) \in Q$ evaluated by a price vector $\pi = (p, \boldsymbol{p}) \in P_+$ is given by $\pi\xi = px + \sum_{t=1}^\infty \boldsymbol{p}^t \boldsymbol{x}^t$.

Let $(A, \mathcal{A}, \lambda)$ be a complete probability space of the consumers. We assume that the consumption set X_∞ that is identical among all consumers is defined by $X_\infty = \mathbb{R}^k_+ \times Z_\infty$, where Z_∞ is a cube in ℓ^∞,

$$Z_\infty = \{\boldsymbol{x} \in L |\ \boldsymbol{0} \leq \boldsymbol{x} \leq \hat{x}\boldsymbol{1}\},$$

where $\boldsymbol{1} = (1, 1 \dots) \in \ell^\infty$ and $\hat{x}$ is a positive constant. Of course, the $\hat{x} > 0$ is intended to be a very large number. By Fact 2.15 in Section 2.2, Z_∞ is a compact metric subspace of L_+ with respect to the weak* topology that is obviously convex.

As usual, a preference $\succsim$ is a complete, transitive, and reflexive binary relation on X_∞ that is close in $X_\infty \times X_\infty$ in the weak* topology. Recall that $\mathcal{P}$ is the set of allowed preferences. Because X_∞ is a locally compact, complete separable metric space, we can show as in Hildenbrand [18, Lemma, p. 98] that $\mathcal{P}$ is a Borel set.

An endowment vector is an element of Q_+. An endowment vector is usually written as $\omega = (e, \boldsymbol{e}) \in Q_+$, $e = (e^i) \in \mathbb{R}^k_+$ and $\boldsymbol{e} = (\boldsymbol{e}^t) \in \ell^\infty_+$. We denote the set of all endowment vectors by Ω and assume that it is of the form

$$\Omega_\infty = \{\omega = (e, \boldsymbol{e}) \in Q |\ \mathbf{0} \leq \omega \leq \hat{\omega}\mathbf{1}\},$$

for some $\hat{\omega} > 0$. The number $\hat{\omega}$ is intended to be far smaller than $\hat{x}$. The set Ω_∞ is also a compact metric space in the weak* topology. The same caveat for X_∞ also applies to Ω_∞. In what follows, we often write for simplicity X_∞ and Ω_∞ as X and Ω, respectively.

The endowment map $a \mapsto \omega(a)$ in the economy $\mathcal{E} : A \to \mathcal{P} \times \Omega$ is Gelfand integrable by Fact 2.18 in Section 2.3. The next assumption saying that the total endowment belongs to the norm interior of the consumption set is standard.

Assumption 5.1. (PE) (Positive endowments). (i) $e(a) > \mathbf{0}$ a.e,
(ii) $\int_A \omega(a) d\lambda \ggg \mathbf{0}$.

We can now state

Theorem 5.2. *Let $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be an economy that is saturated. Then, there exists a competitive equilibrium (π, ξ) for $\mathcal{E}$.*

Theorem 5.3. *Let $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be an economy that satisfies the conditions assumed in Theorem 5.2. A feasible allocation $\xi : A \to X$ belongs to the core of the economy $\mathcal{E}$ if and only if there exists a price vector $\pi \in P$ such that (π, ξ) is a competitive equilibrium for $\mathcal{E}$.*

Remark 5.4. In order to prove Theorem 5.3, the saturation is not required (cf. [51] or [56]). Since the weak* closure of the Gelfand integral $\int_A \Psi(a) d\lambda$ is weak* compact and convex (see [23] or [67]), the Hahn-Banach separation theorem can be still applied with minor modifications. However, if we drop the saturation, non-emptiness of the equilibrium hence the core can be no longer guaranteed, therefore relevance of such a generalization is questionable. Moreover, under the assumption of saturation we can invoke Fact 2.22 which makes the proof simple and straightforward. By these reasons we present Theorem 5.3 assuming the saturation. The same remark applies to Theorem 5.11.

5.2. **A Market with Differentiated Commodities.** Let (K, d_K) be a compact metric space. In this section, we set $L = ca(K)$ and $M = C(K)$; hence, $Q = \mathbb{R}^k \times ca(K)$ and $P = \mathbb{R}^k \times C(K)$ $(k \geq 0)$. A typical element of Q is denoted by $\phi = (u, \boldsymbol{m})$ or $\psi = (y, \boldsymbol{n})$, $u, y \in \mathbb{R}^k$ and $\boldsymbol{m}, \boldsymbol{n} \in ca(K)$. The price vector is written as $\rho = (q, \boldsymbol{q})$, $q = (q^i) \in \mathbb{R}^k$ and $\boldsymbol{q} = q(t) \in C(K)$.

Following [19, 41], the economic interpretation of K is a space of the commodity characteristics. Hence, each $t \in K$ represents the complete list of characteristics that describes the commodity. A (differentiated) commodity bundle $\boldsymbol{m}$ is defined as a signed measure on K; hence, an element of $ca(K)$. In particular, the Dirac measure δ_t is the (one unit of) commodity bundle that contains characteristics $t \in K$. The consumption set $X_\mathcal{M}$ is $X_\mathcal{M} = \mathbb{R}^k_+ \times Z_\mathcal{M}$, $Z_\mathcal{M} = ca(K)_+$. We assume

$k \geq 0$; hence, we do not require the nondifferentiated (or homogeneous) goods. We need an additional assumption for the preferences:

Assumption 5.5. (US) For all $y \in \mathbb{R}^k_+$, for all $\succsim \in \mathcal{P}$ and for every $\alpha > 1$, there exists $\varepsilon > 0$ such that if diameter $(J) < \varepsilon$ and $\boldsymbol{m}, \boldsymbol{n} \in Z_{\mathcal{M}}$ satisfy $\boldsymbol{m}(J) \leq \boldsymbol{n}(J)$, then

$$(y, \boldsymbol{m}_{|J} + \boldsymbol{m}_{|K\setminus J}) \prec (y, \alpha(\boldsymbol{n}_{|J} + \boldsymbol{m}_{|K\setminus J})),$$

where diameter $(J) = \sup\{d(s,t) \mid s,t \in J\}$ and $\boldsymbol{m}_{|J}$ is the restriction of $\boldsymbol{m}$ to J.

The assumption **US** is called by Ostroy and Zame [48] by *uniform substitutability*, meaning that for nearby commodities, consumers prefer any feasible trade in which the "terms" (namely α) are strictly greater than one. Let $\mathcal{P}_{US}$ be the set of preferences that satisfy Assumptions **PR** and **US**.

An initial endowment is assumed to be a nonnegative vector ϖ of $X_{\mathcal{M}}$. We often denote $\varpi = (f, \boldsymbol{f})$, $f = (f^i) \in \mathbb{R}^k_+$ and $\boldsymbol{f} \in ca(K)_+$. Let $\Omega_{\mathcal{M}} \subset X_{\mathcal{M}}$ be the set of initial endowments. We assume it is of the form

$$\Omega_{\mathcal{M}} = \{\varpi = (f, \boldsymbol{f}) \in Q \mid \boldsymbol{0} \leq f^i, \boldsymbol{f}(K) \leq \hat{\omega},\ i = 1 \ldots k\},\ \hat{\omega} > 0.$$

As before the suffix $\mathcal{M}$ is often omitted from $X_{\mathcal{M}}$ and $\Omega_{\mathcal{M}}$.

Let $(A, \mathcal{A}, \lambda)$ be a complete probability space of consumers. An endowment assignment is a Gelfand integrable map $\varpi : A \to \Omega_{\mathcal{M}}$, $a \mapsto \varpi(a) = (f(a), \boldsymbol{f}(a))$. The assumption on the endowments for this economy is

Assumption 5.6. (AE) (Adequate endowments). (i) $\int_A f(a) d\lambda \gg \boldsymbol{0}$,
(ii) *support* $\left(\int_A \boldsymbol{f}(a) d\lambda\right) = K$ (see Fact 2.1).

The assumption **AE** simply says that all commodity characteristics are available in the market. Let $\mathcal{E}_{\mathcal{P}}$ be the composition of the map $\mathcal{E}$ and the projection of $\mathcal{P} \times \Omega$ to $\mathcal{P}$, $a \mapsto \succsim_a$. The existence of equilibria for the model with the differentiated commodities is established by the following:

Theorem 5.7. *Let $\mathcal{E}$ be an economy that is saturated and satisfies $\mathcal{E}_{\mathcal{P}}(A) \subset \mathcal{P}_{US}$ and Assumption* **AE**. *Then, there exists a competitive equilibrium (ρ, ϕ) with $\rho \in P_+ \setminus \{\boldsymbol{0}\}$ for $\mathcal{E}$.*

We say that $\tau (\neq \boldsymbol{0}) \in X$ is an *extremely desirable commodity* if there exists an weak* open neighborhood U of $\boldsymbol{0}$ such that for each $\phi \in X$ we have $\phi \prec \phi + \alpha\tau - \psi$ whenever $\alpha > 0$, $\alpha\tau \geq \psi$, and $\psi \in \alpha U$. This notion is interpretable geometrically as follows. Let $\tau(\neq \boldsymbol{0}) \in X$, U be an weak* open neighborhood U of $\boldsymbol{0}$ and define an open cone C by

$$C = \{\alpha\tau - \psi \mid \psi \in Q, \psi \in \alpha U, \alpha > 0\}.$$

Then, the commodity bundle τ is extremely desirable if for each $\phi \in X$ we have $\phi \prec \chi$ whenever $\chi \in (C + \phi) \cap X$. Note that this implies that τ is an extremely desirable commodity if for each $\phi \in X$ it follows that $((-C + \phi) \cap X) \cap \{\chi \in X \mid \phi \prec \chi\}$ or equivalently $(-C) \cap \{\chi - \phi \in X \mid \phi \prec \chi\} = \emptyset$. Rustichini and Yannelis [56] strengthened the concept of the extremely desirable commodity.

Definition 5.8. A pair of commodities $(\chi, \upsilon) \in X \times X$ is said to be a *desirable commodity pair* if for every $\psi \in X$ we have $\psi \prec_a \psi + \chi - \upsilon$ whenever $\psi + \chi \geq \upsilon$ for each $a \in A$.

Definition 5.9. A pair $(\chi, \upsilon) \in Q \times Q$ is said to have the *splitting property* if for any m-tuple $(\psi_1 \dots \psi_m) \in Q \times \cdots \times Q$ such that $\sum_{i=1}^m \psi_i = (\chi - \upsilon)^-$ there exists an m-tuple $(\gamma_1 \dots \gamma_m) \in Q \times \cdots \times Q$ such that $\sum_{i=1}^m \gamma_i = (\chi - \upsilon)^+$ and the pair (ψ_i, γ_i) is a desirable commodity pair.

The next condition was proposed by [56].

Assumption 5.10. (CD) (Commodity pair desirability). There exists a commodity bundle $\tau(\neq \mathbf{0}) \in X$ and an weak* open neighborhood U of $\mathbf{0}$ such that any commodity bundle pair (χ, υ) of the form $\chi = \alpha\tau$, $\alpha > 0$, and $\upsilon \in \alpha U$ has the splitting property.

Let $\mathcal{P}_{CD}$ be the set of preferences that satisfy the conditions **PR** and **CD**. Let $\mathcal{P}' \subset \mathcal{P}_{CD}$ be a Borel set of allowed preferences. Note that in [19, 41], the sets of allowed preferences are assumed to be compact. Our requirement of measurability is much weaker. The core equivalence theorem for the economy $\mathcal{E}$ on $ca(K)$ is now stated as follows.

Theorem 5.11. *Let $\mathcal{E}$ be an economy that is saturated and satisfies $\mathcal{E}_{\mathcal{P}}(A) \subset \mathcal{P}'$ and Assumption* **AE**. *A feasible allocation $\phi : A \to X$ belongs to the core of the economy $\mathcal{E}$ if and only if there exists a price vector $\rho \in P_+ \backslash \{\mathbf{0}\}$ such that (ρ, ϕ) is a competitive equilibrium for $\mathcal{E}$.*

6. Proofs

6.1. **Proof of Theorem 4.4.** First, we show

Lemma 6.1. *Let $(A, \mathcal{A}, \lambda)$ be an atomless measure space, $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be a representation of μ and $\xi : A \to X$ a measurable mapping. Define $\nu = (\xi, \mathcal{E})_* \lambda$. Then, ξ is an equilibrium allocation of $\mathcal{E}$ if and only if ν is an equilibrium distribution of μ.*

Proof. Suppose that ξ is an equilibrium allocation of $\mathcal{E}$. Then, there exists a price vector $\pi(\neq \mathbf{0}) \in \ell^\infty_+$ with $\lambda(E) = 1$ and $\int_A \xi(a) d\lambda = \int_A \omega(a) d\lambda$, where $E = \{a \in A |\ \pi\xi(a) = \pi\omega(a)$ and $\xi(a) \succsim_a \zeta$ whenever $\pi\zeta \leq \pi\omega(a)\}$. Let

$$F = \{(\xi, \succsim, \omega) \in X \times \mathcal{P} \times \Omega |\ \pi\xi = \pi\omega \text{ and } \xi \succsim \zeta \text{ whenever } \pi\zeta \leq \pi\omega\}.$$

Then, $(\xi, \mathcal{E})^{-1}(F) = E$, hence $\nu(F) = (\xi, \mathcal{E})_* \lambda(F) = \lambda(E) = 1$, which proves the condition (D-1). Because $\xi_* \lambda = \nu_X$ and $\omega_* \lambda = \nu_\Omega = \mu_\Omega$, we have from the change of variable formula $\int_X x d\nu_X = \int_\Omega \omega d\mu_\Omega$. Hence, the condition (D-2) is met. Finally, the condition (D-3) follows from $\nu_{\mathcal{P} \times \Omega} = \mathcal{E}_* \lambda = \mu$. The converse is also proved in a similar way. □

We now prove Theorem 4.4. Because $\mathcal{E}_* \lambda = \mu = \nu_{\mathcal{P} \times \Omega}$, we have from Fact 2.5 in Section 2.1 a measurable map ξ with $\nu = (\xi, \mathcal{E})_* \lambda$. Then, ξ is an equilibrium allocation by Lemma 6.1. □

6.2. **Proof of Theorem 4.7.** To prove Theorem 4.7, we shall prove the following:

Lemma 6.2. *Let $(A, \mathcal{A}, \lambda)$ be an atomless measure space, $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be a representation of μ and ν be a symmetric equilibrium of μ, i.e., there exists a measurable mapping $\theta : \mathcal{P} \times \Omega \to X$ such that $\nu(Graph(\theta)) = 1$. Define $\xi : A \to X$ by $\xi(a) = \theta(\mathcal{E}(a))$. Then, $\nu = (\xi, \mathcal{E})_* \lambda$ and ξ is an equilibrium allocation of $\mathcal{E}$.*

Proof. Let $B \in X \times \mathcal{P} \times \Omega$. Then, we have

$$\begin{aligned}\nu(B \cap Graph(\theta)) &= \nu(\{(\theta(\succsim, \omega), \succsim, \omega) |\ (\theta(\succsim, \omega), \succsim, \omega) \in B\}) \\ &= \mu(\{(\succsim, \omega) |\ (\theta(\succsim, \omega), \succsim, \omega) \in B\}) \\ &= \lambda(\{a \in A |\ (\xi(a), \mathcal{E}(a)) \in B\}) = (\xi, \mathcal{E})_* \lambda(B),\end{aligned}$$

which implies that $\nu = (\xi, \mathcal{E})_* \lambda$ and ξ is an equilibrium allocation of $\mathcal{E}$ from Lemma 6.1. □

Now we turn to prove Theorem 4.7. Let ν be a symmetric equilibrium of the economy μ with a measurable map $\theta : \mathcal{P} \times \Omega \to X$ such that $\nu(Graph(\theta)) = 1$. Set $\xi = \theta \circ \mathcal{E}$. It is easy to verify that ξ is $\sigma(\mathcal{E})$-measurable. It follows from Lemma 6.2 that $\nu = (\xi, \mathcal{E})_* \lambda$ and ξ is an equilibrium allocation of $\mathcal{E}$.

Next, suppose that $\nu = (\xi, \mathcal{E})_* \lambda$ for a $\sigma(\mathcal{E})$-measurable equilibrium allocation of $\mathcal{E}$. By Lemma 6.1, ν is an equilibrium distribution of μ. It follows from Fact 2.3 in Section 2.1 that there exists a measurable map $\theta : \mathcal{P} \times \Omega \to X$ such that $\xi = \theta \circ \mathcal{E}$. It is sufficient to show that $\nu(Graph(h)) = 1$, yielding the following:

$$\begin{aligned}\nu(Graph(\theta)) &= (\xi, \mathcal{E})_* \lambda(Graph(\theta)), \\ &= \lambda(\{a \in A |\ (\xi(a), \mathcal{E}(a)) \in Graph(\theta)\}), \\ &= \lambda(\{a \in A |\ (\theta \circ \mathcal{E}(a), \mathcal{E}(a)) \in Graph(\theta)\}), \\ &= \lambda(\{a \in A |\ \mathcal{E}(a) \in \mathcal{P} \times \Omega\}) = 1,\end{aligned}$$

which proves Theorem 4.7. □

6.3. **Proof of Theorem 4.8.** (a): Let I' be a Borel subset of $I = [0, 1]$ such that $dx(I') = 1$ and $\mathcal{E}$ is one-to one on I'. We can assume without loss of generality that both $\mathcal{E}$ and ξ are constant on $I \backslash I'$. Lemma 6.1 shows that for a competitive equilibrium allocation $\xi : [0, 1] \to X$ of $\mathcal{E}$ that is measurable in $\mathscr{B}([0, 1])$, $\nu = (\xi, \mathcal{E})_* dx$ is an equilibrium distribution of μ. It remains to show that ν is symmetric. Let $B \in \mathcal{P} \times \Omega$ and $C = \xi^{-1}(B) \cap I'$. Because ξ is measurable, $C \in \mathscr{B}([0, 1])$. By Fact 2.2 in Section 2.1, $\mathcal{E}(C) \in \mathscr{B}(\mathcal{P} \times \Omega)$; hence, $C \in \sigma(\mathcal{E})$. Theorem 4.7 implies that ν is symmetric.
(b): Let $\xi : [0, 1] \to X$ be a Gelfand integrable function which is measurable in $\mathscr{B}([0, 1])$, and set $\nu = (\xi, \mathcal{E})_* dx$. By Lemma 6.1, ξ is a competitive equilibrium allocation of $\mathcal{E}$. The proof of the part (a) shows that ξ is $\sigma(\mathcal{E})$ measurable; hence, ν is symmetric.

6.4. **Proofs of Theorem 4.9 and Corollary 4.10.** To prove that (a) implies (b) in Theorem 4.9, let ν be a symmetric equilibrium of an atomless economy μ, and $(A, \mathcal{A}, \lambda)$ an atomless probability space. Then, there exists a measurable map $\theta : \mathcal{P} \times \Omega \to X$ such that $\nu(Graph(\theta)) = 1$. Suppose that $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be a representation of μ. Set $\xi = \theta \circ \mathcal{E}$. Then, it follows from Lemma 6.2 that $\nu = (\xi, \mathcal{E})_* \lambda$ and ξ is an equilibrium allocation. Therefore, (a) implies (b). Obviously, (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Now, suppose that (d) holds. Because μ is atomless, it follows from Fact 2.4 that there exists a representation $\mathcal{E}$ of μ on a measure space $([0, 1], \mathscr{B}([0, 1]), dx)$ which is almost one-to-one. Because $([0, 1], \mathscr{B}([0, 1]), dx)$ realizes ν, there exists an equilibrium allocation ξ of $\mathcal{E}$ such that $\nu = (\xi, \mathcal{E})_* dx$. Then, it follows from Theorem 4.8 that ν is symmetric. □

Finally, we deduce the Corollary. Let $(A, \mathcal{A}, \lambda)$ be saturated. Then, it realizes any distributive equilibrium, i.e., a symmetric equilibrium. Conversely, let ν be nonsymmetric distributive equilibrium of an economy μ. We can easily show that $([0,1], \mathscr{B}([0,1]), dx)$ is a realization of ν if and only if the Lebesgue space $([0,1], \mathcal{L}([0,1]), dx)$ realizes ν. Therefore, by Theorem 4.9, neither $([0,1], \mathscr{B}([0,1]), dx)$ nor $([0,1], \mathcal{L}([0,1]), dx)$ realizes ν. Because $(A, \mathcal{A}, \lambda)$ realizes ν, it follows from Fact 2.10 in Section 2.1 that $(A, \mathcal{A}, \lambda)$ is saturated. □

6.5. **Proof of Theorem 5.2.** Step 1: Let $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be the economy. For each $n \in \mathbb{N}$, let L^n be the canonical projection of ℓ^∞ to $\mathbb{R}^n$, $L^n = \{\xi = (\xi^t) \in \ell^\infty | \ \xi = (\xi^1, \xi^2 \dots \xi^n, 0, 0 \dots)\}$. We then define

$$X^n = X \cap L^n,\ \succsim_a^n = \succsim_a \cap (X^n \times X^n),\ \mathcal{P}^n = \mathcal{P} \cap 2^{L^n \times L^n}, \Omega^n = \Omega \cap L^n.$$

Similarly, we denote the canonical projection of $\omega = (\omega^1, \omega^2 \dots \omega^n, \omega^{n+1} \dots) \in \Omega$ as $\omega_n = (\omega^1, \omega^2 \dots \omega^n, 0, 0 \dots) \in \Omega^n$. Obviously, $\omega_n \to \omega$ in the weak* topology. They induce finite dimensional economies $\mathcal{E}^n : A \to \mathcal{P}^n \times \Omega^n$ defined by $\mathcal{E}^n(a) = (\succsim_a^n, \omega_n(a))$, $n = 1, 2 \dots$. Then, we have the following:

Lemma 6.3. *For each $n \geq k$, there exists a quasicompetitive equilibrium for the economy $\mathcal{E}^n$, or a price-allocation pair $(\pi_n, \xi_n(a))$ that satisfies*
(Q-1n) $\pi_n \xi_n(a) \leq \pi_n \omega(a)$ *and* $\xi_n(a) \succsim_a \zeta$ *whenever* $\pi_n \zeta \leq \pi_n \omega_n(a)$ *and* $\pi_n \omega_n(a) > 0$ *a.e.,*
(Q-2n) $\int_A \xi_n(a) d\lambda \leq \int_A \omega_n(a) d\lambda$.

Proof. See Khan and Yamazaki [34], Proposition 2. □

Step 2: Because $\omega_n(a) = (e_n(a), \boldsymbol{e}_n(a)) \to \omega(a) = (e(a), \boldsymbol{e}(a))$ a.e., we have

$$\int_A \omega_n(a) d\lambda \to \int_A \omega(a) d\lambda$$

by Fact 2.19 in Section 2.3. Without loss of generality, we can assume that $\pi_n \mathbf{1} = \sum_{t=1}^n p_n^t = 1$ for all n, where $\pi_n = (p_n^t)$ and $\mathbf{1} = (1, 1 \dots)$. Here, we have identified $\pi_n \in \mathbb{R}_+^n$ with a vector in ℓ_+^1 that is also denoted by π_n as $\pi_n = (\pi_n, 0, 0 \dots)$.

We denote $\xi_n(a) = (x_n(a), \boldsymbol{x}_n(a)) \in X^n$. The finite dimensional Fatou's lemma (Fact 2.20 in Section 2.3) implies that there exists a measurable map $x : A \to \mathbb{R}_+^k$ such that $x(a) \in Ls(x_n(a))$ a.e. in A and $\int_A x(a) d\lambda \leq \lim_{n\to\infty} \int_A x_n(a) d\lambda$ ($\leq \int_A e(a) d\lambda$). By Fact 2.21 in Section 2.3, we have a Gelfand integrable function $\boldsymbol{x} : A \to X$ such that $\boldsymbol{x}(a) \in Ls(\boldsymbol{x}_n(a))$ a.e., and $\int_A \boldsymbol{x}(a) d\lambda \leq \int_A \boldsymbol{e}(a) d\lambda$. Let $\xi(a) = (x(a), \boldsymbol{x}(a))$. Then we have obtained that

$$\int_A \xi(a) d\lambda \leq \int_A \omega(a) d\lambda.$$

Since the set $\Delta = \{\pi \in ba_+ | \ \|\pi\| = \pi \mathbf{1} = 1\}$ is weak* compact by the Alaoglu's theorem (Fact 2.15 in Section 2.2), we have a subnet $(\pi_{n(\kappa)}, \xi_{n(\kappa)}(a))$ with $(\pi_{n(\kappa)}, \xi_{n(\kappa)}(a)) \to (\hat{\pi}, \xi(a))$ a.e. in the weak* topology, where $\hat{\pi} \in ba_+$ with $\hat{\pi} \mathbf{1} = 1$.

Step 3: Define the set $P = \{a \in A|\ \hat{\pi}\omega(a) > 0\}$. Because $\pi_n \in \ell^1$ for all n, we have $\int_A \pi_{n(\kappa)}\omega(a)d\lambda = \pi_{n(\kappa)} \int_A \omega(a)d\lambda \to \hat{\pi} \int_A \omega(a)d\lambda > 0$ by Assumption **PE** (ii) and $\hat{\pi}\mathbf{1} = 1$; hence, we obtain $\lambda(P) > 0$. The essence of the proof is contained in the next Lemma.

Lemma 6.4. *$\xi(a) \prec_a \zeta$ implies that $\hat{\pi}\omega(a) < \hat{\pi}\zeta$ a.e. on P.*

Proof. If the lemma was false, there exists $\zeta(a) = (\zeta^t(a)) \in X$ such that $\hat{\pi}\zeta(a) \leq \hat{\pi}\omega(a)$ and $\xi(a) \prec_a \zeta(a)$ on a subset of P with λ-positive measure. Let $Q_n = \{a \in A|\ a$ does not satisfy (Q-1n)$\}$ and $R = \{a \in A|\ \xi(a) \notin Ls(\xi_n(a))\}$. Set $Q = \cup_{n=1}^{\infty} Q_n$. Since $Q \cup R$ is of measure 0, $P\backslash(Q \cup R)$ is nonempty. Let $a \in P\backslash(Q \cup R)$. We can assume without loss of generality that $\hat{\pi}\zeta(a) < \hat{\pi}\omega(a)$ and $\xi(a) \prec_a \zeta(a)$. Let $\zeta_n(a) = (\zeta^1(a) \dots \zeta^n(a), 0, 0 \dots)$ be the projection of $\zeta(a)$ to X^n. Because $\zeta_n(a) \to \zeta(a)$, we have for sufficiently large n_0 that $\hat{\pi}\zeta_{n_0}(a) \leq \hat{\pi}\zeta(a) < \hat{\pi}\omega(a)$ and $\xi(a) \prec_a \zeta_{n_0}(a)$.

Because $(\pi_{n(\kappa)}, \xi_{n(\kappa)}) \to (\hat{\pi}, \xi(a))$, there is a κ_1 with $n(\kappa_1) \equiv n_1 \geq n_0$ such that $0 \leq \pi_{n_1}\zeta_{n_0}(a) < \pi_{n_1}\omega(a) = \pi_{n_1}\omega_{n_1}(a)$, and $\xi_{n_1}(a) \prec_a \zeta_{n_0}(a)$, or $\xi_{n_1}(a) \prec_a^{n_1} \zeta_{n_0}(a)$. This contradicts the fact that $(\pi_{n_1}, \xi_{n_1}(a))$ is a quasi-equilibrium for $\mathcal{E}^{n_1}$. □

By Lemma 6.4 and Assumption **PR** (ii) we obtain $p \gg \mathbf{0}$; hence, it follows from Assumption **PE** (i) that $\lambda(P) = 1$. Let $\hat{\pi} = \pi + \pi_p$ be the Yosida–Hewitt decomposition where $\pi \in \ell^1_+$ is the countably additive part and π_p is purely the finitely additive part. Lemma 6.4 combined with Assumption **PR** (ii) imply that $\pi\xi(a) \geq \pi\omega(a)$ a.e. Then, it follows from the resource feasibility condition that we have the budget conditions $\pi\xi(a) = \pi\omega(a)$ for almost all $a \in A$. The condition (E-1) follows immediately from Lemma 6.4.

Because $\int_A \boldsymbol{x}^t(a)d\lambda \leq \int_A \boldsymbol{e}^t(a)d\lambda \leq \hat{\omega} < \hat{x}$ for each t, there exists a positive amount of consumers with $\boldsymbol{x}^t(a) < \hat{x}$. Then, by the monotonicity **PR** (ii), one obtains that $\boldsymbol{p}^t > 0$ for all t. Because $p \gg \mathbf{0}$, it follows that $\int_A \xi(a)d\lambda = \int_A \omega(a)d\lambda$, or the condition (E-2) is met. This completes the proof of Theorem 5.2. □

6.6. **Proof of Theorem 5.3.** The proof that a competitive equilibrium allocation is a core allocation is standard. Therefore, it is skipped.

Step 1: We shall show $\mathscr{C}(\mathcal{E}) \subset \mathscr{W}(\mathcal{E})$. Let $\phi(\cdot) \in \mathscr{C}(\mathcal{E})$ and define $P : A \to \ell^\infty$ by $P(a) = \{\xi \in \ell^\infty|\ \phi(a) \prec_a (\xi + \omega(a))\}$ and $\Psi : A \to \ell^\infty$ by $\Psi(a) = P(a) \cup \{\mathbf{0}\}$, respectively. It follows from Assumption **PR** (ii) and unboundedness of the first k goods that $P(a) \neq \emptyset$, a.e.

Lemma 6.5. *$Graph(\Psi) \in \mathscr{B}(A \times \ell^\infty)$.*

Proof. By the measurability of $\mathcal{P}$ and the continuity of preferences, the set

$$G = \{(\prec, \xi, \eta) \in \mathcal{P} \times \ell^\infty \times \ell^\infty|\ \eta \prec \xi\}$$

is a Borel set of $\mathcal{P} \times \ell^\infty \times \ell^\infty$. Defining a map $\psi : A \times \ell^\infty \to \mathcal{P} \times \ell^\infty \times \ell^\infty$ by $\psi(a, \xi) = (\prec_a, \xi + \omega(a), \phi(a))$, it follows that $Graph(P) = \psi^{-1}(G)$. Because the map ψ is measurable by Remark 2.23 of Section 2.3, the graph of Ψ is measurable. □

Because $\Psi(a)$ is weak* measurable by Lemma 6.5 and Remark 2.23, we can define the integral $\int_A \Psi(a)d\lambda$. Because $\mathbf{0} \in \int_A \Psi(a)d\lambda$, it is nonempty and convex by Fact 2.22. We now show that $\int_A \Psi(a)d\lambda \cap \ell^\infty_- = \{\mathbf{0}\}$. Suppose not. Then, there exists a function $h : A \to \ell^\infty$ such that $h(a) \in \Psi(a)$ a.e., and $\int_A h(a)d\lambda < \mathbf{0}$. Set $C = \{a \in A|\ h(a) \neq \mathbf{0}\}$. Obviously, $\lambda(C) > 0$. Define an integrable function $g : A \to \ell^\infty$ by

$$g(a) = h(a) + \omega(a) - \frac{\int_A h(a)d\lambda}{\lambda(C)}.$$

We can write $g(a) = (\tilde{g}(a), \boldsymbol{g}(a)) = ((\tilde{g}^i(a)), (g^t(a))) \in \mathbb{R}^k \times \ell^\infty$, and let $\hat{\boldsymbol{g}}(a) = (\hat{g}^t(a))$ by $\hat{g}^t(a) = \inf\{g^t(a), \hat{x}\}$. Setting $\hat{g}(a) = (\tilde{g}(a), \hat{\boldsymbol{g}}(a))$, it can be easily seen that $\phi(a) \prec_a \hat{g}(a)$ a.e. in C and $\int_C \hat{g}(a)d\lambda = \int_C \omega(a)d\lambda$, contradicting $\phi \in \mathscr{C}(\mathcal{E})$. We can apply Fact 2.16 to the two disjoint convex sets $\int_A \Psi(a)d\lambda$ and $\ell^\infty_-\backslash\{\mathbf{0}\}$ and obtain a vector $\hat{\pi} \in ba$ with $\hat{\pi} > \mathbf{0}$ such that

$$0 \leq \hat{\pi}\zeta \text{ whenever } \zeta \in \int_A \Psi(a)d\lambda.$$

Step 2: We shall show that

$$\hat{\pi}\eta \geq \hat{\pi}\omega(a) \text{ whenever } \phi(a) \prec_a \eta \text{ a.e.}$$

To do this, we first verify $\hat{\pi}\phi(a) = \hat{\pi}\omega(a)$ a.e. in A. Let $C \subset A$ be a measurable set with $\lambda(C) > 0$ and take an $\epsilon > 0$ and $\boldsymbol{d} \in \mathbb{R}^k_+\backslash\mathbf{0}$. We define $j : A \to X$ by the following

$$j(a) = \begin{cases} \phi(a) - \omega(a) + \epsilon(\boldsymbol{d}, \mathbf{0}) & \text{for } a \in C, \\ 0 & \text{for } a \notin C. \end{cases}$$

Then, it follows from Assumption **PR** (ii) that $j(a) \in \Psi(a)$; hence,

$$\hat{\pi}\left(\int_C \phi(a) + \epsilon\lambda(C)(\boldsymbol{d}, \mathbf{0}) - \int_C \omega(a)d\lambda\right) \geq 0,$$

and rearranging this, we obtain $\int_C \hat{\pi}\phi(a)d\lambda \geq \int_C \hat{\pi}\omega(a)d\lambda - \epsilon\lambda(C)\hat{\pi}(\boldsymbol{d}, \mathbf{0})$; hence, $\int_C \hat{\pi}\phi(a)d\lambda \geq \int_C \hat{\pi}\omega(a)d\lambda$ for any $C \subset A$, because $\epsilon > 0$ is arbitrary. We then conclude $\hat{\pi}\phi(a) \geq \hat{\pi}\omega(a)$ a.e. It follows from $\int_A \phi(a)d\lambda \leq \int_A \omega(a)d\lambda$ that $\hat{\pi}\phi(a) = \hat{\pi}\omega(a)$ a.e., as desired. By replacing $\phi(a)$ in the definition of $j(a)$ by η with $\phi(a) \prec_a \eta$ for $a \in C$, we obtain $\int_C \hat{\pi}\eta d\lambda \geq \int_C \hat{\pi}\omega(a)d\lambda$. Because $C \subset A$ is arbitrary, we conclude $\hat{\pi}\eta \geq \hat{\pi}\omega(a)$ for every η such that $\phi(a) \prec_a \eta$ a.e. in A.

The remaining part of the proof proceeds in the same way as in the last part of the proof of Theorem 5.2. Let $\hat{\pi} = \pi_c + \pi_p$ be the Yosida–Hewitt decomposition where $\pi_c \equiv (p, \boldsymbol{q}) \in \mathbb{R}^k_+ \times \ell^1_+$ is the countably additive part and π_p is purely finitely additive part. The budget conditions $\pi_c\phi(a) = \pi_c\omega(a)$ a.e. can be shown as usual. The above condition and Assumption **PR** (ii) imply $p \gg \mathbf{0}$, hence $\pi_c\omega(a) > 0$ a.e. by Assumption **PE**. We then conclude $\phi(a) \prec_a \eta$ implies $\pi_c\eta > \pi_c\omega_a$, a.e. This completes the proof. □

6.7. **Proof of Theorem 5.7.** Step 1: Let ϵ_n be a sequence of positive numbers decreasing to zero. As in [19, 41], we can construct a sequence of finite subsets $K^n = \{t^n_1 \dots t^n_{m_n}\}$ of K and a sequence of pairwise disjoint open sets B^n_i with

$t_i^n \in B_i^n$ for $i = 1 \dots m_n$ such that denoting $B^n = \cup_{i=1}^{m_n} B_i^n$,

$$d_K(t_i^n, t) \leq \epsilon_n \text{ for every } t \in B_i^n \text{ and for all } n, i = 1 \dots m_n,$$
$$\int_A \varpi(a) d\lambda(B^n) = \int_A \varpi(a) d\lambda(K),$$
$$K^n \subset K^{n+1} \text{ for all } n, \text{ and } K^n \to K \text{ in the topology of closed convergence.}$$

For each n, let $L^n = LS(t_1^n \dots t_{m_n}^n) \subset ca(K)$ be the linear space spanned by $\{\delta_{t_1^n} \dots \delta_{t_{m_n}^n}\}$ and set $X^n = \mathbb{R}^k \times (Z \cap L^n)$. We then define $\mathcal{P}^n = \mathcal{P} \cap (X^n \times X^n)$ and let r^n be a map from $\mathcal{P}$ to $\mathcal{P}^n$ defined by

$$r^n(\succsim) = \succsim^n = \succsim \cap (X^n \times X^n).$$

Jones [19, Lemma 6] proved that the map $r^n(\cdot)$ is continuous. It is obvious that $r^n(\succsim) \to \succsim$ as $n \to +\infty$.

For each n, let $\Lambda^n : \mathcal{P} \times X \to \mathcal{P}^n \times X^n$ be a map defined by

$$\Lambda^n(\succsim, \varpi) = (\succsim^n, \varpi_n), \; \succsim^n = r^n(\succsim), \; f_n = f, \; \boldsymbol{f}_n = \sum_{i=1}^{m_n} \boldsymbol{f}(B_i^n) \delta_{t_i^n}.$$

It is obvious from definition that $\varpi_n(K) = \varpi(K)$. Because r^n is continuous and B_i^n are open, the map Λ^n is measurable. Set $\mathcal{E}^n = \Lambda^n \circ \mathcal{E} : A \to \mathcal{P}^n \times X^n$, $\mathcal{E}^n(a) = (\succsim_a^n, \varpi_n(a))$. Then, $\mathcal{E}^n$ is an economy with finite number of commodities. Because *support* $\left(\int_A \boldsymbol{f}(a) d\lambda\right) = K$, we have by construction

$$\int_A \varpi_n(a) d\lambda \gg \mathbf{0},$$

where we have identified X^n with $\mathbb{R}_+^{k+m_n}$. Then, we have the following

Lemma 6.6. *The economy $\mathcal{E}^n$ has a competitive equilibrium, or there exists a price vector $\rho_n \in \mathbb{R}_+^{k+m_n}$ with $\rho_n \neq \mathbf{0}$ and an allocation $\phi_n : A \to X^n$ that satisfy*
(E-1n) $\rho_n \phi_n(a) \leq \rho_n \varpi_n(a)$ and $\phi_n(a) \succsim_a \psi$ whenever $\rho_n \psi \leq \rho_n \varpi_n(a)$ a.e. in A,
(E-2n) $\int_A \phi_n(a) d\lambda = \int_A \varpi_n(a) d\lambda$.

Proof. Because $\int_A \varpi_n(a) d\lambda \gg \mathbf{0}$ and the preferences are monotone, the assumptions for Theorem 2 in Hildenbrand [18, p. 151] are satisfied. $\square$

Step 2: We denote $\rho_n = (q_n, \boldsymbol{q}_n) = ((q_n^i), (q_n(t))) \in \mathbb{R}_+^k \times \mathbb{R}_+^{m_n}$. Without loss of generality, we can assume $\|\rho_n\| = \max\{q_n^i, q_n(t) | \; i = 1 \dots k, \; t \in K^n\} = 1$ for every n. In the next lemma, the assumption of bounded marginal rate of substitution **US** plays an essential role.

Lemma 6.7. *Let (ρ_n, ϕ_n) be the equilibrium obtained by Lemma 6.6. Then, $(K^n, \boldsymbol{q}_n)$ are equicontinuous.*

Proof. Suppose that $(K^n, \boldsymbol{q}_n)$ are not equicontinuous. Then, we can assume that there exist sequences (t_n), (s_n) such that

$$d(t_n, s_n) \to 0 \text{ and } \lim_{n \to \infty} \frac{q_n(t_n)}{q_n(s_n)} > 1.$$

For t_n with $\boldsymbol{m}_n(\{t_n\}) > 0$, define

$$\boldsymbol{m}_n^{(t_n,s_n)} = \boldsymbol{m}_n - \boldsymbol{m}_n(\{t_n\})\delta_{t_n} + \left(\frac{q_n(t_n)}{q_n(s_n)}\right)\boldsymbol{m}_n(\{t_n\})\delta_{s_n}.$$

Because $\int_A \boldsymbol{m}_n(a)d\mu = \int_A \boldsymbol{e}_n(a)d\mu \gg \boldsymbol{0}$, we have

$$\mu\left(\{a \in A|\ \boldsymbol{m}_n(a)(t_n) > 0\}\right) > 0.$$

If $\boldsymbol{m}_n(a)(t_n) > 0$, then by Assumption **US**, we have for n sufficiently large that

$$(u_n(a), \boldsymbol{m}_n(a)) \prec_a (u_n(a), \boldsymbol{m}_n^{(t_n,s_n)}(a)).$$

This is a contradiction because $\boldsymbol{q}_n\boldsymbol{m}_n^{(t_n,s_n)}(a) = \boldsymbol{q}_n\boldsymbol{m}_n(a)$. □

We can assume without loss of generality that $q_n \to q \in \mathbb{R}^k_+$, and it follows from Fact 2.13 in Section 2.1 that we can assume that $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$ for some $\boldsymbol{q} \in C(K)$. Let $\rho_n = (q_n, \boldsymbol{q}_n)$ and $\rho = (q, \boldsymbol{q})$. Clearly $\|\rho\| = 1$.

Lemma 6.8. *Suppose that $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$ and $\varpi_n \to \varpi \in \Omega$ with $\rho\varpi > 0$. Let $\phi_n(a) = (u_n(a), \boldsymbol{m}_n(a)) \in \mathbb{R}^k_+ \times \mathbb{R}^{m_n}_+$ and suppose $u_n(a) \to u(a)$, and for each n, $\rho_n\phi_n(a) \leq \rho_n\varpi_n$ and $\psi \precsim^n_a \phi_n(a)$ whenever $\rho_n\psi \leq \rho_n\varpi_n$. Then, if $\boldsymbol{q}(t^*) = 0$ for some $t^* \in K$, then $\boldsymbol{m}_n(a)(K) \to +\infty$.*

Proof. Suppose not. Then, by taking subsequence if necessary, we can assume that $\boldsymbol{m}_n(a) \to \hat{\boldsymbol{m}}$ for some $\hat{\boldsymbol{m}} \in Z$. We now claim that $\rho\phi(a) \leq \rho\varpi(a)$ and $\phi(a) \succsim_a \psi$ whenever $\rho\psi \leq \rho\varpi(a)$, where $\phi(a) = (u(a), \hat{\boldsymbol{m}})$ and $\psi = (y, \boldsymbol{n})$. To see this, suppose that there exists $\psi \in X$ with $\rho\psi \leq \rho\varpi(a)$ and $\phi(a) \prec_a \psi$. Because $\rho\varpi(a) > 0$ we can assume without loss of generality that $\rho\psi < \rho\varpi(a)$ and $\phi(a) \prec_a \psi$. Setting $\psi_n = (y, \boldsymbol{n}_n)$, where $\boldsymbol{n}_n = \sum_{i=1}^{m_n} \boldsymbol{n}(B^n_i)\delta_{t^n_i}$, we have $\psi_n \to \psi$; hence, $\rho_n\psi_n < \rho_n\varpi_n(a)$ and $\phi_n(a) \prec^n_a \psi_n$ for n large enough, contradicting the assumption. This cannot be the case, however, because $\boldsymbol{q}(\boldsymbol{m} + \delta_{t^*}) = \boldsymbol{q}\boldsymbol{m}$ and $(u(a), \hat{\boldsymbol{m}}) \prec_a (u(a), \hat{\boldsymbol{m}} + \delta_{t^*})$. □

Step 3: We now claim that there exists an $\epsilon > 0$ such that $q_n(t) \geq \epsilon > 0$ for all $t \in K^n$, $n = 1, 2 \ldots$. If this is not the case, there exists a sequence $\{t^n\} \subset K^n$ such that $t^n \to t^*$ for some $t^* \in K$ with $q_n(t^n) \to 0$. Obviously, this implies that $q(t^*) = 0$ as well. Choose $s \in K$ such that $q(s) = 1/2$ and take an open neighborhood U of s such that $q(t) > 1/2$ for $t \in U$. Let $B = \{a \in A|\ \varpi(a)(U) > 0\}$. Clearly, B is measurable, and by Assumption **AE**, $\lambda(B) > 0$. Because $\boldsymbol{q}_n\boldsymbol{f}_n(a) \to \boldsymbol{q}\boldsymbol{f}(a) > 0$ a.e. on B by Fact 2.12 in Section 2.2, we have $\rho_n\varpi_n(a) \to \rho\varpi(a) > 0$ a.e. on B, and it follows from Lemma 6.8 that $\int_A \boldsymbol{f}_n(a)d\lambda(K) = \int_A \boldsymbol{m}_n(a)d\lambda(K) \geq \int_B \boldsymbol{m}_n(a)d\lambda(K) \to +\infty$. However, this contradicts that

$$\int_A \boldsymbol{f}_n(a)d\lambda(K) \to \int_A \boldsymbol{f}(a)d\lambda(K) < +\infty.$$

Because $\boldsymbol{q}_n \geq \epsilon$ and $\|\varpi(a)\| \leq \hat{\omega}$, it follows that

$$0 \leq \boldsymbol{m}_n(a)(K) \leq \frac{\hat{\omega}}{\epsilon} \text{ for all } n \text{ large enough, a.e.}$$

Then, by applying Fact 2.21 in Section 2.3, we obtain that there exists a measurable map $\boldsymbol{m}(a)$ such that

$$\int_A \boldsymbol{m}(a)d\lambda \in Ls\left(\int_A \boldsymbol{m}_n(a)d\lambda\right) \text{ and } \boldsymbol{m}(a) \in Ls(x_n(a)) \text{ a.e.}$$

It follows from

$$\lim_{n\to\infty}\int_A u_n(a)d\lambda \le \lim_{n\to\infty}\int_A f_n(a)d\lambda = \lim_{n\to\infty}\int_A f(a)d\lambda < +\infty$$

and Fact 2.20 that there exists a measurable map $u : A \to \mathbb{R}^k_+$ such that $u(a) \in Ls(u_n(a))$ a.e. in A and $\int_A u(a)d\lambda \le \lim_{n\to\infty}\int_A u_n(a)d\lambda$. Let $\phi(a) = (u(a), \boldsymbol{m}(a))$. We show that $(\rho, \phi(\cdot))$ is an equilibrium for $\mathcal{E}$.

Because $\int_A \varpi_n(a)d\lambda = \int_A \phi_n(a)d\lambda$, $\phi_n(a) \to \phi(a)$ a.e. and $\varpi_n(a) \to \varpi(a)$ a.e., it follows from Fact 2.19 in Section 2.3 that

$$\int_A \varpi(a)d\lambda = \lim_n \int_A \varpi_n(a)d\lambda = \lim_n \int_A \phi_n(a)d\lambda = \int_A \phi(a)d\lambda,$$

and one obtains from $\rho_n \to \rho$ and Fact 2.12 in Section 2.2 that

$$\rho\phi(a) \le \rho\varpi(a) \text{ a.e.}$$

Hence, the condition (E-2) and the budget conditions are met.

Suppose that there exists $\psi = (y, \boldsymbol{n}) \in X$ such that $\rho\psi \le \rho\varpi(a)$ and $\phi(a) \prec_a \psi$. If $\rho\varpi(a) > 0$, then because the preferences are continuous, we can assume without loss of generality that $\rho\psi < \rho\varpi(a)$ and $\phi(a) \prec_a \psi$. Let $\psi_n = (y, \sum_{i=1}^{m_n} \boldsymbol{n}(B_i^n)\delta_{t_i^n})$. Because $\rho_n\varpi_n(a) \to \rho\varpi(a)$ and $\rho_n\psi_n \to \rho\psi$, it follows from (E-1n) and (6.2) that $\rho_n\psi_n < \rho_n\varpi_n(a)$ and $\phi_n(a) \prec_a \psi_n$ for n large enough, a contradiction. If $\rho\varpi(a) = 0$, we can easily show from Assumptions **PR** (ii) and **AE** that $q \gg \mathbf{0}$. Then, because $q(t) \ge \epsilon$ for all $t \in K$, the budget set is a singleton, or $\{\phi \in X |\ \rho\phi \le 0\} = \{\mathbf{0}\}$; hence, $\phi(a) = \mathbf{0}$ is trivially a maximal element in the budget set. Therefore, the condition (E-1) is also met and we complete the proof. □

6.8. **Proof of Theorem 5.11.** Step 1: In the following proof, we omit the prime of $\mathcal{P}'$ and denote it simply as $\mathcal{P}$. The proof that a competitive equilibrium allocation is also a core allocation is standard. Therefore, it is skipped. For the opposite direction, let $\phi : A \to X$ be a core allocation, and we need the following auxiliary result.

Lemma 6.9. *Define the upper contour set of* $\phi(a)$ *by*

$$\Phi_{\phi(a)} = \{\gamma \in X \mid \phi(a) \prec_a \gamma\}.$$

Then, the correspondence $a \mapsto \Phi_{\phi(a)}$ *has a measurable graph.*

Proof. By the measurability of $\mathcal{P}$ and the continuity of preferences, the set $G = \{(\precsim, \psi, \gamma) \in \mathcal{P}\times X\times X \mid \psi \prec \gamma\}$ belongs to $\mathscr{B}(\mathcal{P}\times X\times X)$. Define $\Gamma : A\times X \to \mathcal{P}\times X\times X$ by $\Gamma(a, \gamma) = (\precsim_a, \phi(a), \gamma)$. Obviously, Γ is a measurable map by the remark 2.23; hence,

$$\text{Graph}\,(\Phi_{\phi(a)}) = \Gamma^{-1}(G) \in \mathscr{B}(A\times X)$$

. □

We show that there exists a price vector $\rho \in P$ such that (ρ, ϕ) is a competitive equilibrium for $\mathcal{E}$. Define the correspondence $\Psi : A \to 2^X$ by

$$\Psi(a) = \Phi_{\phi(a)} \cup \{\varpi(a)\}.$$

Step 2: Let $C = \cup_{\alpha>0}\alpha(\tau - U)$, where $\tau(\neq \mathbf{0}) \in X$ and U are given in Assumption **CD**. We show that

$$\left(\int_A \Psi(a)d\lambda - \int \varpi(a)d\lambda\right) \cap -C = \emptyset.$$

Because $-C$ is weak* open, it is sufficient to show that for any Gelfand integrable function $\psi : A \to X$ such that $\int_A \psi(a)d\lambda \in \int_A \Psi(a)d\lambda$, and there exists a sequences of Gelfand integrable maps $\psi_n : A \to X$ and $\varpi_n : A \to X$ such that $\psi_n(a) \to \psi(a)$ and $\varpi_n(a) \to \varpi(a)$ a.e. in the weak* topology, hence $\int_A \psi_n(a)d\lambda \to \int_A \psi(a)d\lambda$ and $\int_A \varpi_n(a)d\lambda \to \int_A \varpi(a)d\lambda$ and

$$\int_A \psi_n(a) - \int_A \varpi_n(a)d\lambda \notin -C$$

for a large enough n. Let $S = \{a \in A | \ \phi(a) \prec_a \psi(a)\}$. If $\lambda(S) = 0$, then $\psi(a) = \varpi(a)$ a.e. by definition of $\Psi(a)$; hence, $\int_A \psi(a)d\lambda - \int_A \varpi(a)d\lambda \notin -C$. Hence, we may assume that $\lambda(S) > 0$.

For each $n \in \mathbb{N}$, define $S_n = \{a \in S | \ \|\psi(a)\|(= \psi(a)(K)) \leq n\}$. Obviously, $S_1 \subset S_2 \subset \dots$ are measurable and $\cup_{n=1}^{\infty} S_n = S$, $\lambda(S_n) \to \lambda(S)$ as $n \to \infty$. Hence, we can take a measurable subset S_N of S such that $\lambda(S_N)$ is arbitrarily close to $\lambda(S)$ and $\psi(a)$ is norm bounded on S_N. If we restrict our discussion on S_N, then we can assume without loss of generality that $\{\psi_n(a)\}$ are included in a bounded subset $\hat{X} = \{\phi \in X | \ \|\phi\| \leq N + 1\}$. On $\hat{X}$, the weak* topology is metrizable by Alaoglu's theorem (Fact 2.15 in Section 2.1); hence, in the same way as [59, Lemma 7], we have a sequence of simple functions $\hat{\psi}_n : S \to X$ with $\hat{\psi}_n(a) \to \psi(a)$ a.e. in $\hat{S}_n$ in the weak* topology. Then, by Egorov's Theorem (Fact 2.11 in Section 2.1), there exists a measurable set $S' \subset S_N$ of positive measure with $\lambda(S')$ arbitrarily close to $\lambda(S_N)$ such that $\hat{\psi}_n(a) \to \psi(a)$ uniformly on S'. By the continuity of preferences, for sufficiently large n, we have $\phi(a) \prec_a \hat{\psi}_n(a)$ a.e. in S'.

We can write

$$\hat{\psi}_n(a) = \sum_{i=1}^{m_n} \hat{\psi}_n^i \mathbf{1}_{S_n^i}(a),$$

where

$$\phi(a) \prec_a \hat{\psi}_n^i \text{ for all } a \in S_n^i \text{ and all } i = 1 \dots m_n,$$

and $\mathbf{1}_{S_n^i}(\cdot)$ is the indicator function of the set S_n^i with $S' = \cup_{i=1}^{m_n} S_n^i$. Without loss of generality, we may assume that $\lambda(S_n^1) = \cdots = \lambda(S_n^{m_n}) = \hat{\lambda}$. Let $\varpi_n(a) = \sum_{i=1}^{m_n} \left(\hat{\lambda}^{-1} \int_{S_n^i} \varpi(a)d\lambda\right) \mathbf{1}_{S_n^i} = \sum_{i=1}^{m_n} \varpi_n^i \mathbf{1}_{S_n^i}$.

We now show that $\int_{S'} \hat{\psi}_n(a) - \int_{S'} \varpi_n(a)d\lambda \notin -C$. Suppose not. Then, for some $\alpha > 0$ one has

$$\sum_{i=1}^{m_n} \hat{\psi}_n^i \hat{\lambda} - \sum_{i=1}^{m_n} \varpi_n^i \hat{\lambda} \in -\alpha(\tau + U)$$

and therefore

$$\sum_{i=1}^{m_n} \hat{\psi}_n^i + \chi - \upsilon = \sum_{i=1}^{m_n} \varpi_n^i,$$

where $\chi = (\alpha/\hat{\lambda})\tau$ and $\upsilon \in (\alpha/\hat{\lambda})U$. Because $\sum_{i=1}^{m_n} \varpi_n^i \geq \mathbf{0}$, it follows from the presented equation that

$$(\chi - \upsilon)^- \leq \sum_{i=1}^{m_n} \hat{\psi}_n^i.$$

Then, by the Riesz decomposition theorem (Fact 2.17 in Section 2.2), we can find $\mathbf{0} \leq \psi_n^i \in Q$ with $\mathbf{0} \leq \psi_n^i \leq \hat{\psi}_n^i$, $i = 1 \ldots m_n$ and $\sum_{i=1}^{m_n} \psi_n^i = (\chi - \upsilon)^-$. It follows from Assumption **CD** that there exist m_n commodity bundles $\gamma_n^1 \ldots \gamma_n^{m_n}$ such that $\sum_{i=1}^{m_n} \gamma_n^i = (\chi - \upsilon)^+$ and

$$\hat{\psi}_n^i \prec_a \hat{\psi}_n^i + \gamma_n^i - \psi_n^i \text{ for all } a \in S_n^i \text{ and all } i = 1 \ldots m_n.$$

Set $\tilde{\psi}_n^i = \hat{\psi}_n^i + \gamma_n^i - \psi_n^i$. We have $\phi(a) \prec_a \tilde{\psi}_n^i$ from $\phi(a) \prec_a \hat{\psi}_n^i$ for all $a \in S_n^i$ and all i and the transitivity of $\prec_a$. Define $\tilde{\psi}_n(a) = \sum_{i=1}^{m_n} \tilde{\psi}_n^i \mathbf{1}_{S_n^i}(a)$. Then,

$$\int_{S'} \tilde{\psi}_n(a) d\lambda = \sum_{i=1}^{m_n} \tilde{\psi}_n^i \hat{\lambda} = \sum_{i=1}^{m_n} \varpi_n^i \hat{\lambda} = \int_{S'} \varpi(a) d\lambda.$$

We have found an allocation $\tilde{\psi}_n(a)$ that is feasible for the coalition S' and preferred to $\phi(a)$ a.e. in S', which contradicts the assumption that $\phi(a)$ is a core allocation.

Define $\psi_n : A \to X$ by

$$\psi_n(a) = \begin{cases} \hat{\psi}_n(a) & \text{for } a \in S', \\ \psi(a) & \text{for } a \notin S'. \end{cases}$$

Similarly, we define $\varpi_n : A \to X$ by

$$\varpi_n(a) = \begin{cases} \varpi_n(a) & \text{for } a \in S', \\ \varpi(a) & \text{for } a \notin S'. \end{cases}$$

Because $S' \subset S$ and $\lambda(S')$ is arbitrarily close to $\lambda(S)$ and C is weak* open, we conclude that $\int_A \psi_n(a) - \int_A \varpi_n(a) d\lambda \notin -C$; hence,

$$\left(\int_A \phi(a) d\lambda - \int \varpi(a) d\lambda \right) \cap -C = \emptyset.$$

Step 3: By Fact 2.22 in Section 2.3, $\int_A \Psi(a) d\lambda$ is convex. By the Hahn–Banach separation theorem (Fact 2.16 in Section 2.2), there exists $\rho \in P$ such that

$$\rho\psi \geq \rho \int_A \varpi(a) d\lambda$$

for every $\psi \in \int_A \Psi(a) d\lambda$. The monotonicity **PR** (ii) implies that $\rho \geq \mathbf{0}$.

We will show that for a.e.

$$\psi \succ_a \phi(a) \text{ implies that } \rho\psi \geq \rho\varpi(a).$$

To do this, first we now show that $\rho\phi(a) = \rho\varpi(a)$ a.e. Let $C \subset A$, $\lambda(C) > 0$ and $\epsilon > 0$. Define $\hat{j} : A \to X$ by

$$\hat{j}(a) = \begin{cases} \phi(a) + \epsilon \int_A \varpi(a) d\lambda & \text{for } a \in C, \\ \varpi(a) & \text{for } a \notin C. \end{cases}$$

Then, we have

$$\rho \left(\int_C \phi(a) d\lambda + \epsilon\lambda(C) \int_A \varpi(a) d\lambda + \int_{A \setminus C} \varpi(a) \right) \geq \rho \int_A \varpi(a) d\lambda.$$

Given that $\epsilon > 0$ is arbitrary, we rearrange this inequality to have that $\int_C \rho\phi(a)d\lambda \geq \int_C \rho\varpi(a)d\lambda$ for any $C \subset A$. It follows that $\rho\phi(a) \geq \rho\varpi(a)$ a.e. Because $\int_A \phi(a)d\lambda = \int_A \varpi(a)d\lambda$, we have $\rho\phi(a) = \rho\varpi(a)$ a.e. Replacing in the definition of $\hat{j}(a)$ by $\psi \in \Psi(a)$ for $a \in C$, we have $\int_C \rho\psi d\lambda \geq \int_C \rho\varpi(a)d\lambda$. Because $C \subset A$ is arbitrary, we conclude that

$$\psi \succ_a \phi(a) \text{ implies that } \rho\psi \geq \rho\varpi(a) \text{ a.e.}$$

The following argument is standard. For $a \in A$ with $\rho\varpi(a) > 0$, we can show that $\rho\psi > \rho\varpi(a)$ whenever $\phi(a) \prec_a \psi$. By Assumption **PR** (ii), we can easily show that ρ is strictly positive. Then, for $a \in A$ such that $\rho\varpi(a) = 0$, the budget set is a singleton $\{\mathbf{0}\}$, meaning that $\phi(a) = \mathbf{0}$ is utility maximizing under the budget constraint; hence, condition (E-1) of Definition 5.8 holds. This shows that (ρ, ϕ) is a competitive equilibrium. □

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